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Analysis of cosmic structures with Minkowski tensors



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Analyse kosmischer Strukturen mit Minkowskitensoren

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Abstract

Modern cosmological research heavily relies on statistical analyses of the shape of large cosmic structures. Tracers range from temperature variations in the cosmic microwave background to the distribution of galaxies and galaxy clusters. One common aspect of all these data sets is their distribution over the whole sky, calling for analysis tools that can be applied on the surface of the sphere. Additionally, observational data is plagued by foregrounds — the cosmic microwave background may be covered by anything else in the observable universe — which requires rigorous masking to be handled by said tools.

One promising avenue is provided by the Minkowski functionals and tensors. These are versatile morphometric instruments from integral geometry and characterize basic geometric properties of shapes such as contour length and preferred directions. They comprehensively describe all additive shape information for convex shapes. Masking can be handled straightforwardly as the tools operate in real space, as opposed to, e.g., the power spectrum. On the sphere, the functionals and the Cartesian representation of the tensors have been previously established. In this work, I have developed a version of the irreducible Minkowski tensors on the sphere, enabling simple access to higher symmetries and the corresponding preferred directions. Additionally, I provide a publicly available software package for the calculation of both irreducible and Cartesian Minkowski tensors on the sphere depicted as maps for data in the commonly used HEALPix format.

I then apply a localized Minkowski analysis to cosmic microwave background data and simulations presented by the Planck Collaboration, finding two notable spots where the temperature fluctuations in the data appear more elongated than in the simulations. Furthermore, I showcase the usefulness of Minkowski tensors in analyzing a plethora of large-scale structure data coming from vastly different sources and data formats, specifically the Uchuu and GLAM simulations and a selection of SDSS data, the Planck cluster catalog, eROSITA all-sky maps, the eROSITA eRASS 1 galaxy group and cluster catalog, and Magneticum simulations showing slices of structure formation at different redshifts. For the latter, the Minkowski tensors are sensitive to differences in maps generated for different cosmic parameters. This selection demonstrates the versatility of the tools at hand and provides a handbook for future users for both noisy and smoothed data. The Minkowski functionals and tensors in both representations are now readily available to the scientific community, with possible applications reaching beyond cosmology and astronomy.

Kurzzusammenfassung

Die moderne Forschung im Feld der Kosmologie beruht zu großen Teilen auf statistischen Analysen der Form großräumiger kosmischer Strukturen. Diese sind durch den kosmischen Mikrowellenhintergrund und die Verteilung von Galaxien und Galaxienhaufen gegeben. All diesen Datensätzen gemein ist eine weiträumige Verteilung über den gesamten Himmel, die nach Analysewerkzeugen auf der Kugeloberfläche verlangt. Weiterhin sind solche Daten stark von Vordergrundquellen verunreinigt — der kosmische Mikrowellenhintergrund kann von allem anderen im Universum verdeckt sein. Dadurch ist rigoroses Maskieren notwendig, das von den besagten Werkzeugen miteinbezogen werden muss.

Ein vielversprechender Ansatz sind die Minkowskifunktionale und -tensoren. Das sind vielseitige morphometrische Werkzeuge aus der Integralgeometrie, die grundlegende geometrische Eigenschaften von Formen wie Konturlänge und Ausrichtung charakterisieren. Für konvexe Formen beinhalten sie sämtliche additive Information. Maskierung der Daten kann direkt miteinbezogen werden, da sie im realen Raum definiert sind, im Gegensatz zu beispielsweise dem Leistungsspektrum. Auf der Sphäre waren die Funktionale und die sogenannte kartesische Darstellung bereits etabliert. In dieser Arbeit habe ich eine Version der sogen. irreduziblen Darstellung auf der Kugeloberfläche entwickelt, die einen einfachen Zugang zu höheren Symmetrien und den zugehörigen Vorzugsrichtungen ermöglichen. Zusätzlich stelle ich ein öffentlich zugängliches Softwarepaket für die Berechnung sowohl irreduzibler als auch kartesischer Minkowskitensoren für Daten auf der Kugel im verbreiteten HEALPix-Format zur Verfügung.

Ich wende sodann eine lokale Minkowskianalyse auf Daten und Simulationen des kosmischen Mikrowellenhintergrundes der Planck Collaboration an, wo zwei Flecken hervortreten, an denen die Strukturen in den Temperaturdaten länglicher wirken als in den Simulationen. Weiterhin zeige ich den Nutzen der Minkowskitensoren in der Analyse von Daten der großräumigen Struktur aus unterschiedlichsten Quellen und Dateiformaten, gegeben durch die Uchuu und GLAM Simulationen und eine Selektion von SDSS Daten, den Planck cluster Katalog, eROSITA Himmelskarten, den eROSITA eRASS1 Katalog von Galaxiengruppen und -haufen sowie Magneticum-Simulationen von Ausschnitten der Strukturentstehung bei verschiedenen Rotverschiebungen. In letzterem Datensatz sind die Minkowskitensoren auf Unterschiede zwischen Karten empfindlich, die für unterschiedliche kosmische Parameter erzeugt wurden. Diese Auswahl verdeutlicht die Vielseitigkeit der gezeigten Werkzeuge und fungiert als ein Nutzerhandbuch für zukünftige Anwender für geglättete und verrauschte Daten. Die Minkowskifunktionale und -tensoren in beiden Darstellungen stehen somit der wissenschaftlichen Gemeinschaft zur Verfügung, wobei mögliche Anwendungen über Astronomie und Kosmologie hinaus gehen.

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Chapter 1

Background: Cosmology

This work will go into the morphometric analysis of the very largest cosmic structures in the form of Cosmic Microwave Background (CMB) temperature fluctuations and the large-scale distribution of galaxies and galaxy clusters. Such analyses may serve to explore cosmic structure formation, conditions in the early universe, or cosmic parameters, e.g. by comparison to simulation data. Notably, all these data sets comprise structures from large regions of the sky, necessitating measures that operate on the surface of the sphere. The tools must also be able to handle masking due to extended foregrounds such as the Galactic Plane. The target of this thesis is to introduce and demonstrate the use of powerful morphometric tools (Minkowski functionals and Minkowski tensors) on the sphere, which are capable of intuitively handling such data and contain comprehensive structure information. While they will be discussed in chapter 2 and the analyses performed are shown after that, it makes sense to first acquaint oneself with the physical concepts and questions standing behind modern cosmological research. The reader is invited to settle down for the experience, food and drink recommendations will follow along the way. For a good start, I suggest crêpe with butter and lemon, possibly sugar to taste.

The following chapter gives an introduction to the basics of cosmology and the cosmic standard model. I restrict myself to those the ideas and concepts that are necessary for understanding the CMB, inflation, and Large-Scale Structure (LSS) as these topics will be addressed in the subsequent analyses. I will thereafter also give a short overview of recent issues in cosmology.

In particular, I follow the outline and definitions of Bartelmann, 2019. Other works (see e.g. Odintsov et al., 2023; Bartolo et al., 2004 for reviews of inflation) may use slightly different (but equivalent) parameters. Further, I will use the convention $(-, +, +, +)$ for the sign of the diagonal elements of the spacetime metric.

1.1 The cosmic standard model

1.1.1 Metric and basic considerations

At the basis of the cosmic standard model lie two assumptions of symmetry: The observed properties of the universe should, averaging over large enough scales, be the same in any direction (isotropy) and this should hold in any location; no position in the

universe is special (homogeneity). Furthermore, it is assumed that the laws of general relativity govern cosmic behavior.

In general the line element for a metric g is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \quad (1.1)$$

Introducing comoving coordinates associated with fundamental observers that do not change with changing length scales ($dx^i = 0$)

$$ds^2 = g_{00} dt^2 = -c^2 dt^2 \quad (1.2)$$

has to hold, where c is the vacuum speed of light.

Isotropy implies that $g_{0i} = 0$ as there can be no preferred direction, so the line element is given by

$$ds^2 = -c^2 dt^2 + g_{ij} dx^i dx^j . \quad (1.3)$$

Such a spacetime can be separated into spacelike hypersurfaces of constant cosmic time each. Then the spacelike part of the line element must be defined by only a scale factor $a(t)$ in order to preserve isotropy and homogeneity.

There may, however, still be a non-zero spacelike curvature K present giving the spatial line element $dl^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{(1+Kr^2/4)^2}$. Negative curvature refers to hyperbolic space, positive to 3-spheres.

Introducing $dw = \frac{dr}{1+Kr^2/4}$ one gets

$$dl^2 = dw^2 + f_K^2(w)[d\theta^2 + \sin^2 \theta d\phi^2] = dw^2 + f_K^2(w)d\Omega^2 \quad (1.4)$$

with

$$f_K(w) = \begin{cases} K^{-1/2} \sin(K^{1/2}w) & (K > 0) \\ w & (K = 0) \\ |K|^{-1/2} \sinh(|K|^{1/2}w) & (K < 0) \end{cases}$$

This gives rise to the Robertson-Walker (RW) metric with line element

$$ds^2 = -c^2 dt^2 + a^2(t)(dw^2 + f_K^2(w)d\Omega^2) . \quad (1.5)$$

Redshift

Photons moving through an expanding or contracting universe as governed by $a(t)$ will be red- or blueshifted. To calculate this frequency shift, consider light ($ds = 0 \Rightarrow c|dt| = a(t)dw$) emitted at t_e and observed at t_o for comoving source and observer. That is, both are moving with the cosmic flow and their coordinate distance is constant:

$$w_{eo} = \int_{t_e}^{t_o} dw = \int_{t_e}^{t_o} \frac{cdt}{a(t)} = \text{const.} . \quad (1.6)$$

Thus, its derivative by t_e must vanish and one gets

$$\frac{dt_o}{dt_e} = \frac{a(t_o)}{a(t_e)} , \quad (1.7)$$

or written in terms of frequency or wavelength and following the definition of kinematic redshift

$$\frac{\nu_e}{\nu_o} = \frac{a(t_o)}{a(t_e)} = \frac{\lambda_o}{\lambda_e} = 1 + \frac{\lambda_o - \lambda_e}{\lambda_e} =: 1 + z \quad (1.8)$$

Since astronomical data is usually observed at $a(t_o) = 1$ one writes $a(t_e) = a$ and gets

$$a = \frac{1}{1 + z} \quad (1.9)$$

1.1.2 The Friedmann equations

In order to consider the dynamics of such a spacetime one uses the Einstein field equations

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} - \Lambda g_{\alpha\beta} \quad (1.10)$$

with cosmological constant Λ , gravitational constant G , Einstein tensor $G_{\alpha\beta}$ (calculated from the Riemann tensor, in turn coming from the metric tensor and its derivatives), and the energy-momentum tensor $T_{\alpha\beta}$.

The Einstein tensor is diagonal in this case, so the same must hold for the energy momentum tensor. Its 00 component is then given by the energy density ρc^2 and its spatial elements need to be equal to the pressure P due to isotropy. This is the energy-momentum tensor of an ideal fluid. Due to homogeneity, pressure and density may only depend on time.

Applying all of the above to the RW metric yields the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3} \quad (1.11)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} . \quad (1.12)$$

A metric whose scale factor satisfies these equations is called a Friedmann-Lemaître-Robertson-Walker (FLRW) metric. By convention one sets the current scale factor $a(t_0) = a_0 = 1$, where the index zero is used for the current value of a parameter.

Combining them yields the equation of adiabatic energy conservation:

$$\frac{d}{dt}(a^3 \rho c^2) + P \frac{d}{dt}(a^3) = 0 . \quad (1.13)$$

One may now introduce the equation-of-state-parameter

$$w := \frac{P}{\rho c^2} , \quad (1.14)$$

which depends on the type of matter considered. Inserting it into the adiabatic equation gives

$$\frac{d}{dt}(a^3 \rho c^2) + w \rho c^2 \frac{d}{dt}(a^3) = 0 \quad (1.15)$$

or

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a} . \quad (1.16)$$

Solving this equation for ρ , one gets the density in terms of the scale factor:

$$\rho(a) = \rho_0 a^{-3(1+w)} . \quad (1.17)$$

For non-relativistic matter (also referred to as dust) the kinetic energy of the particles is far below their rest energy, so one sets $P = 0 = w$. Here, the density $\rho_m \sim a^{-3}$ decays purely due to thinning of the particles in the expanding space. In the relativistic case it is known that $P = \rho c^2/3$, so that $w = 1/3$. The density $\rho_r \sim a^{-4}$ decays faster than the purely geometric expansion of space due to redshift draining energy from the particles.

A special case to be considered here is $w = -1$, yielding a constant density related to the cosmological constant, which will come up again in sec. 1.3. Its pressure is equal to its negative density which in turn is constant over time. In general, $w < 0$ makes the energy density change slower than the geometric expectation, and in the extreme case of phantom equations of state ($w < -1$), the density rises with expansion.

It is now convenient to introduce the Hubble parameter describing the rate of expansion

$$H := \frac{\dot{a}}{a} . \quad (1.18)$$

Furthermore one defines the critical density as

$$\rho_{\text{cr}}(t) \equiv \frac{3H^2(t)}{8\pi G} . \quad (1.19)$$

When $\rho = \rho_{\text{cr}}$ and in absence of a cosmological constant, the universe is flat ($K = 0$); otherwise a positive or negative curvature must be present. It corresponds to roughly one galaxy per Mpc^3 .

In order to rewrite the Friedmann equations it makes sense to introduce the density parameters

$$\Omega(t) = \frac{\rho(t)}{\rho_{\text{cr}}(t)} \quad (1.20)$$

for matter,

$$\Omega_{\Lambda}(t) = \frac{\Lambda c^2}{3H^2(t)} \quad (1.21)$$

for the cosmological constant, and

$$\Omega_K(t) = -\frac{Kc^2}{H^2(t)} \quad (1.22)$$

for curvature.

The density parameters for different types of matter at current time may be separated:

$$\Omega_{\text{r},0} = \frac{\rho_{\text{r},0}}{\rho_{\text{cr},0}} , \quad \Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{\rho_{\text{cr},0}} \quad (1.23)$$

and then inserted into eq. 1.17 to obtain the full scale-factor dependencies

$$\rho_{\text{r}}(a) = \Omega_{\text{r},0} \rho_{\text{cr},0} a^{-4} , \quad \rho_{\text{m}}(a) = \Omega_{\text{m},0} \rho_{\text{cr},0} a^{-3} . \quad (1.24)$$

Inserting these into Friedmann I (1.11), one gets the most commonly used form of this Friedmann equation:

$$H^2(a) = H_0^2 \left(\Omega_{\text{r},0} a^{-4} + \Omega_{\text{m},0} a^{-3} + \Omega_{\Lambda,0} + \Omega_{K,0} a^{-2} \right) =: H_0^2 E^2(a) , \quad (1.25)$$

where $E(a)$ is called the expansion function. Nowadays $H^2(a_0 = 1) := H_0$, so $E(1) \stackrel{!}{=} 1$.

1.1.2.1 Time dependency of parameters

The density parameters evolve dependent on the scale factor. From the above equations one gets for radiation and non-relativistic matter

$$\Omega_r(a) = \frac{8\pi G}{3H^2(a)} \rho_{r,0} a^{-4} = \frac{\Omega_{r,0} a^{-4}}{E^2(a)}, \quad \Omega_m(a) = \frac{\Omega_{m,0} a^{-3}}{E^2(a)}, \quad (1.26)$$

and for the cosmological constant density parameter

$$\Omega_\Lambda(a) = \frac{\Lambda c^2}{3H^2(a)} = \frac{\Omega_{\Lambda,0}}{E^2(a)}. \quad (1.27)$$

When any of these parameters dominate the expansion function, their density parameter at that time or scale factor goes to unity. In particular in the early universe (low a) the density parameter for the cosmological constant quickly drops to zero due to rising contributions from radiation and matter.

The relation between scale factor and time can also be derived via the expansion function by noticing the relation

$$\frac{da}{dt} = aH = aH_0 E(a) \quad (1.28)$$

and solving either for $t(a)$ or $a(t)$. In the radiation dominated era ($E(a) \sim a^{-2}$) $a(t) \sim \sqrt{t}$, $a(t) \sim t^{2/3}$ in the non-relativistic era (called the Einstein-de Sitter case), and $a(t) \sim \exp(t)$ when the cosmological constant dominates in the late universe (de Sitter-case). The latter accelerated expansion is usually treated as being caused by a substance with the given negative pressure, dubbed dark energy.

1.1.2.2 Parameter values

Recent measurements have shown that the current radiation density parameter stemming from the CMB and neutrinos is close to zero ($\sim 10^{-5}$) and negligible for current dynamics. For the Hubble parameter and the other density parameters, the Planck Collaboration VI, 2020 have found $H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_{m,0} = 0.315 \pm 0.007$, and $\Omega_{K,0} = 0.0007 \pm 0.0019$. Sometimes, the Hubble parameter is given as the dimensionless h with $H_0 = h \cdot 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$. It should be noted that there are tensions between different approaches to the measurement of H_0 (see sec. 1.6.2).

Using (1.24) it is possible to find the scale factor of equal radiation and matter densities:

$$\rho_m(a_{\text{eq}}) \stackrel{!}{=} \rho_r(a_{\text{eq}}) \quad (1.29)$$

$$\Omega_{m,0} \cdot a_{\text{eq}} = \Omega_{r,0} \quad (1.30)$$

$$a_{\text{eq}} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \quad (1.31)$$

Planck Collaboration VI, 2020 find $z_{\text{eq}} \sim 3402 \pm 26$ corresponding to $a_{\text{eq}} = (2.94 \pm 0.03) \cdot 10^{-4}$, before which the universe was radiation dominated.

1.1.3 Distances and horizons

Measuring distances cannot be done straightforwardly since the scale factor changes during the measurement. Several types of cosmological distances are commonly used and are presented here.

Proper distance

An intuitive definition is the so-called proper distance: It is defined by the time light takes from emission to observation

$$dD_{\text{prop}} = -cdt = -c \frac{1}{\dot{a}} da = -\frac{c}{H_0 a E(a)} da . \quad (1.32)$$

Comoving distance

Besides that, there is the comoving distance describing the distance on a spacelike hypersurface at constant time, measured in coordinates of the mean cosmic flow. As such, it is the coordinate distance $dD_{\text{com}} = dw$:

$$ds = 0 \Rightarrow adw = -cdt , \quad dw = -c \frac{da}{a\dot{a}} = -c \frac{da}{H_0 a^2 E(a)} \quad (1.33)$$

Angular distance

The definition of the angular distance relies on the relation between the solid angle and the area it would cover on a sphere in Euclidean space with radius D_{ang} :

$$\delta\Omega D_{\text{ang}}^2 = \delta A \quad (1.34)$$

$$D_{\text{ang}} = \sqrt{\frac{\delta A}{\delta\Omega}} \quad (1.35)$$

The ratio of the area covered to the area of the whole sphere must scale like the solid angle to 4π . The total area is not only affected by the scale factors between source and observer, but also by $f_K(w)$.

In other words, the angular distance is the radius of a sphere in Euclidean space where a given solid angle covers the same area fraction as in FLRW spacetime. It shrinks for large redshifts since due to cosmic expansion structures in the early universe were closer together, reducing the area covered by a solid angle. It is given by

$$D_{\text{ang}} = \frac{a_2}{a_1} f_K(w(z_1, z_2)) , \quad (1.36)$$

where $z_1 < z_2$ and the corresponding $a_1 > a_2$.

Luminosity distance

Luminosity distance deals with the ratio of the emitted vs. observed photon flux. Redshift decreases photon energy, thins out their density as compared to Euclidean space, and delays their arrival times. This leads to a reduction of $(a_1/a_2)^4$ in flux, which absorbed into the distance gives

$$D_{\text{lum}}(z_1, z_2) = \left(\frac{a_1}{a_2}\right)^2 D_{\text{ang}}(z_1, z_2) . \quad (1.37)$$

This is the so-called Etherington relation.

Hubble-Lemaître law

At low redshift, all distances above converge. This can be seen by approximating $a(z) = 1/(1+z) \simeq 1 - z$ which gives

$$D = \frac{cz}{H_0} + \mathcal{O}(z^2) . \quad (1.38)$$

cz can then be treated as an apparent velocity rising linearly over distance. This is the Hubble-Lemaître law

$$v = H_0 D . \quad (1.39)$$

Horizons

Consider light traveling over a distance between times t_1 and t_2 :

$$\Delta w(t_1, t_2) = \int_{t_1}^{t_2} \frac{cdt}{a(t)} = c \int_{a_1}^{a_2} \frac{da}{a\dot{a}} . \quad (1.40)$$

At early times (t and a going to zero) the cosmological constant and curvature terms are negligible in Friedmann I (1.11). Assuming the density follows a power law in a with index $-n$, it can be shown that

$$\Delta w(t_1, t_2) = \frac{c}{H_0 \sqrt{\Omega_0}} \int_{a_1}^{a_2} \frac{da}{a^{2-n/2}} = \frac{2c}{(2-n)H_0 \sqrt{\Omega_0}} \left(a_1^{n/2-1} - a_2^{n/2-1} \right) . \quad (1.41)$$

This distance diverges for $n < 2$ when $a_1 \rightarrow 0$ (however in this case, the curvature term may not be negligible). Otherwise, the distance remains finite, meaning that a particle at time t_2 may only have been influenced by events closer than $\Delta w(0, t_2)$. For this reason, this concept is called the particle horizon.

Another, simpler, quantity often encountered in the literature is the time-dependent Hubble radius:

$$r_H = \frac{c}{H(t)} \quad (1.42)$$

Furthermore, assuming $a \sim t^m$ for late times when expansion is dominated by the cosmological constant, then $\Delta w(a, \infty) \sim t^{1-m}$, which converges for $m > 1$. This is the (finite) event horizon encompassing everything that can ever be seen by the observer:

$$r_{\text{eh}} = \Delta w(t, \infty) . \quad (1.43)$$

It is conceptually related to backwards light cone describing the observable universe for an observer at time t_0 given by

$$r_{\text{lc}} = c(t_0 - t) = c \int_t^{t_0} dt' = c \int_a^1 \frac{dx}{x^2 E(x)} . \quad (1.44)$$

It gives the radius from which light emitted at time t has reached the observer. Due to cosmic expansion, it shrinks for early emission times.

1.2 Thermodynamics and recombination

Having discussed the more geometric aspects of cosmic evolution, the focus will now move on to the radiation and matter populating the universe, i.e. thermodynamics. This section will present some general ideas about freeze-out and number densities. Contents of a standard lecture on statistical physics are not explicitly introduced here, the reader is pointed to either chapter 3 of Bartelmann, 2019 or any standard resource on thermodynamics.

It is usually assumed that the changing scale factor corresponds to an adiabatic transition since energy transfer between regions would violate homogeneity. Expansion shall occur slow enough for thermal equilibrium at any time (interaction rates must be quicker than cosmic expansion). If a particle species cannot maintain thermal equilibrium in later times, one says it freezes out. Additionally, all components are assumed to behave like (possibly relativistic) ideal gases without long-range interactions. The latter means that oppositely charged populations must shield each other's interaction range.

For an ideal non-relativistic gas expanding adiabatically it is known that

$$\frac{P}{P_0} = \left(\frac{V}{V_0} \right)^{-\gamma} \quad \text{and} \quad (1.45)$$

$$\frac{T}{T_0} = \left(\frac{V}{V_0} \right)^{1-\gamma} = a^{3(1-\gamma)} , \quad (1.46)$$

where the volume scales with a^3 . The adiabatic index γ for non-relativistic matter is calculated from the particle degrees of freedom

$$\gamma = \frac{f+2}{f} . \quad (1.47)$$

For a one-atomic, non-relativistic ideal gas $\gamma = 5/3$, and for relativistic quantum gases $\gamma = 4/3$.

This enables the calculation of photon temperature as a function of scale factor as

$$T = \frac{T_0}{a} , \quad (1.48)$$

and for a non-relativistic gas one gets

$$T = \frac{T_0}{a^2} , \quad (1.49)$$

so non-relativistic matter cools down faster than photons.

Considering again the early universe with

$$\dot{a} = a \sqrt{\frac{8\pi G}{3} \rho} , \quad (1.50)$$

one can calculate the expansion time scale as $t_{\text{exp}} = 1/H = a/\dot{a} \sim \rho^{-1/2}$. Since for radiation $\rho_r \sim a^{-4}$, the time scale rises with $t_{\text{exp}} \sim a^2$. This means that the expansion slows down over time.

This is the time scale on which thermal equilibrium must be reached. In the early universe creation and annihilation is a main form of interaction between components.

It is governed by the collision rate $\Gamma := dN/dt$ with the number of collisions per time interval

$$dN = n\langle\sigma v\rangle dt . \quad (1.51)$$

Here, n is the particle density, and $\langle\sigma v\rangle$ is the mean value of the product of cross section $\sigma(v)$ and velocity v .

This means

$$\Gamma = n\langle\sigma v\rangle \sim n \sim T^3 \sim a^{-3} , \quad (1.52)$$

so the ratio of collision and expansion timescales $t_{\text{exp}}/t_{\text{coll}} = 1/a$. Early on, this ratio takes very high values; thermal equilibrium is easily maintained, and later over time, the interaction rate drops off and thermal equilibrium can no longer be kept up.

Consider now the continuity equation with collisions and thermal particle creation

$$\dot{n} + \nabla(n\vec{v}) = -\Gamma n + S , \quad (1.53)$$

where $S = n_T^2\langle\sigma v\rangle$ is the thermal creation rate. Inserting furthermore the Hubble-Lemaître law $\vec{v} = H\vec{r}$ with $\nabla\vec{r} = 3$ the continuity equation becomes

$$\dot{n} + 3Hn = -\Gamma \left(1 - \frac{n_T^2}{n^2}\right) . \quad (1.54)$$

Replacing n with the comoving number density $N := a^3n$ and transforming the time derivative $d/dt = \dot{a}d/da = aH d/da = H d/d\ln a$ leads to the useful form

$$\frac{d\ln N}{d\ln a} = -\frac{\Gamma}{H} \left(1 - \frac{N_T^2}{N^2}\right) . \quad (1.55)$$

From this equation several conclusions can be drawn. First, if at any point $N = N_T$, then the comoving number density remains unchanged as expected. If the number density is above (below) its thermal value, the right hand side turns negative (positive) and the number density falls (rises) to the thermal value. The rate of adaptation $\Gamma/H = t_{\text{exp}}/t_{\text{coll}}$ determines if thermal equilibrium can be reached fast enough with respect to expansion. If not, the population *freezes out*; particles are unable to maintain thermal equilibrium.

For relativistic particles, the equilibrium density $n_T \sim a^{-3}$, meaning N_T is constant. Then, if at any point they reach thermal equilibrium $d\ln N/d\ln a = 0$ and both equilibrium and actual number densities remain constant, even if $t_{\text{coll}} \gg t_{\text{exp}}$. Relativistic particles keep their comoving number density from the moment of freezing out from equilibrium.

In the non-relativistic case, the thermal equilibrium density $N_T \sim T^{-3/2} e^{-mc^2/k_B T}$ and drops exponentially once the temperature falls below the rest energy. Then, $d\ln N/d\ln a \simeq \frac{-\Gamma}{H} \rightarrow 0$ and the comoving particle density remains constant.

The number density of baryons today (mainly hydrogen) can be calculated from the density parameter as

$$n_b = \frac{\rho_{B,0}}{m_p} = \frac{\Omega_{B,0} \cdot 3H_0^2}{m_p \cdot 8\pi G} \simeq 5.11 \cdot 10^{-6} \Omega_{B,0} \text{ cm}^{-3} \quad (1.56)$$

with Planck H_0 and $\Omega_{B,0}h^2 = 0.02237 \pm 0.00015$ (Planck Collaboration VI, 2020). For photons as a bosonic gas one gets the number density from the CMB temperature using

$$n_{\gamma,0} = g_\gamma \frac{\zeta(3)}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \simeq 405 \text{ cm}^{-3} , \quad (1.57)$$

where $g_\gamma = 2$ and ζ is the Riemann zeta function.

Both of these densities scale equally with temperature and scale factor ($n \sim T^3 \sim a^{-3}$), so their ratio remains constant. Putting in the numbers shows that $\eta := n_B/n_\gamma \sim 1/1.5 \cdot 10^9$.

It should be noted that a non-negligible amount of relativistic density is contributed by neutrinos, but they freeze out early once their weak coupling to mainly electrons and positrons drops off. From here on it will thus be assumed that the cosmic temperature is controlled by photons.

1.2.1 Origin of the CMB

As the universe cools down, at some point electrons and protons recombine to form hydrogen. The equilibrium in the corresponding reaction



shifts to the right hand side for falling photon energy. Note that here, H stands for hydrogen, and not the Hubble parameter.

In order to study this in detail, consider the Helmholtz free energy for the canonical partition function Z_c

$$F(T, V, N) = -k_B T \ln Z_c . \quad (1.59)$$

The reaction is assumed to take place in a photon bath defining the temperature. Photons are thus not part of the partition function in this case, which is then given by

$$Z_c = \frac{Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}}{N_e! N_p! N_H!} , \quad (1.60)$$

where N_i are particle numbers of protons, electrons, and hydrogen, respectively, and Z_i are the corresponding partition functions for a single particle.

Using $N_e = N_p$, $N_H = N_B - N_e$ and one can express every number in terms of N_B and N_e , giving for the free energy

$$F(T, V, N) = -k_B T \cdot [N_e(\ln Z_e + \ln Z_p) + (N_B - N_e) \ln Z_H - 2 \ln N_e! - \ln(N_B - N_e)!] . \quad (1.61)$$

In order to get the equilibrium ionization from here, one needs to minimize F with respect to the number of electrons. Also, since N_i are very large, the Stirling formula may be applied to simplify the factorials as $\ln N! \simeq N \ln N - N$, $\frac{d}{dN} \ln N! = \ln N$.

The resulting relations between number and partition functions are given by the law of mass action

$$0 \stackrel{!}{=} \frac{\partial F}{\partial N_e} = -k_B T (\ln Z_e + \ln Z_p - \ln Z_H - 2 \ln N_e - \ln(N_B - N_e)) \quad (1.62)$$

$$\Rightarrow \frac{Z_e Z_p}{Z_H} = \frac{N_e^2}{N_B - N_e} . \quad (1.63)$$

The partition function for a single, free, non-relativistic particle is given by

$$Z = \frac{gV}{2\pi^2 \hbar^3} \int_0^\infty dp p^2 \exp(-\beta(\varepsilon(p) - \mu)) , \quad (1.64)$$

where g is the multiplicity of each state ($g = 2$ for photons and massive particles of spin $1/2$ and $g = 1$ for hydrogen in ground state), $\beta = 1/k_B T$, $\varepsilon(p) \simeq mc^2 + \frac{p^2}{2m}$ is the particle's energy in terms of momentum, and μ is the chemical potential.

Chemical potential and mass are independent of momentum, so Z ends up as

$$Z = \frac{gV}{2\pi^2\hbar^3} e^{\beta(\mu-mc^2)} \int_0^\infty dp p^2 \exp\left(-\beta \frac{p^2}{2m}\right). \quad (1.65)$$

Substituting $y^2 = \beta \frac{p^2}{2m}$ and solving the integral leads to

$$Z = gV \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{\beta(\mu-mc^2)}. \quad (1.66)$$

This term, specialized for the three particle species, can now be entered into (1.63). The sum of the chemical potentials $\mu_e + \mu_p - \mu_H$ must be zero in equilibrium, and the mass difference between both sides of the reaction must be the ionization energy of hydrogen $(m_e + m_p - m_H)c^2 = 13.6 \text{ eV} = \chi$. Furthermore, the ionization fraction $x := N_e/N_B$ is introduced since this will be the quantity of interest.

The result is the Saha equation

$$\frac{g_e g_p}{g_H} m_e^{3/2} \left(\frac{k_B T}{2\pi\hbar^2} \right)^{3/2} \frac{\exp(-\beta\chi)}{n_B} = \frac{x^2}{1-x} \quad (1.67)$$

describing the ionization fraction in terms of temperature.

Inserting known relations and values and substituting $n_B = \eta \cdot n_\gamma$ gives

$$\frac{x^2}{1-x} = \frac{\sqrt{\pi}}{\sqrt{2}\zeta(3)} \left(\frac{m_e c^2}{k_B T} \right)^{3/2} \frac{\exp(-\beta\chi)}{\eta} =: f(T). \quad (1.68)$$

The ionization fraction reaches 0.5 at $f(T_{\text{rec}}) = 0.5$, corresponding to a temperature of $T_{\text{rec}} \simeq 3600 \text{ K}$. This corresponds to a scale factor $a_{\text{rec}} = \frac{T_0}{T_{\text{rec}}} \simeq 7.5 \cdot 10^{-4}$ and a redshift $z_{\text{rec}} \simeq 1330$.

Recombination takes finite time, so it may be interesting to see the corresponding temperature range. Between $x = 0.1$ and 0.9 , $T \in [3300, 3900] \text{ K}$, corresponding to a redshift interval of ~ 200 between early and late recombination. One therefore refers to a recombination shell of finite thickness.

These temperatures are far below the ionization energy of hydrogen due to the overwhelming number of thermal photons.

As a sidenote, the Saha equation is not entirely accurate as a description of recombination. Thermal equilibrium is not strictly given: the mean free path of electrons and protons increases rapidly, and the recombination reaction usually releases Lyman α photons that support the reionization of the newly formed hydrogen. These factors slow the process down somewhat with respect to the Saha equation.

In order to estimate the origin of the current CMB photons, consider a photon last scattering within dl and then passing through a layer of thickness l without another scattering. Dividing l into N slices of equal thickness, the probability of this process is given by

$$dP_N(l) = n_e \sigma_T dl \left(1 - n_e \sigma_T \frac{l}{N} \right)^N, \quad (1.69)$$

where n_e is the electron density and σ_T is the Thomson scattering cross section. As $N \rightarrow \infty$, the right hand side of the equation goes to $d\tau e^{-\tau}$ with optical depth $d\tau = n_e \sigma_T dl$.

The relation between dl and the redshift interval dz is given by

$$dl = c dt = \frac{c da}{\dot{a}} = \frac{c da}{H_0 a E(a)} = -\frac{c}{H_0} \frac{a dz}{E(a)}. \quad (1.70)$$

From here, one can get $dP(z)$ and thus $dP(z)/dz$. The resulting distribution has a wide peak of FWHM $\sigma_z \sim 130$ in z . Since $T = T_0(1+z)$ this width corresponds to a temperature difference of $\delta T \simeq T_0 \sigma_z \simeq 350$ K.

The subsequent CMB photons, however, have equal temperature no matter when they last scattered. As the plasma temperature is governed by photons up to recombination, its temperature follows the same scale factor dependency $T \sim 1/a$ as the scattered photons. The photons last scattering earlier in hotter plasma get redshifted more after scattering, by exactly the same amount as the plasma cools down simultaneously.

One speaks of decoupling between radiation and matter when the optical depth τ drops below 1. This occurs at a redshift of about 1250, at a temperature of ~ 3400 K. The CMB as observable today has a temperature of about 2.7260 ± 0.0013 K (Fixsen, 2009).

A few historic notes

First speculations that radiation of early cosmic origin may still be observable today were made by Lemaître, 1931, although he suggested cosmic rays as candidates. Tolman, 1934a showed that radiation with a blackbody spectrum will keep a blackbody spectrum throughout adiabatic cosmic expansion. A cosmic microwave background radiation was then first estimated to be observable at a temperature of around 5 K by Alpher and Herman, 1948. It was finally discovered by Penzias and Wilson, 1965, who detected a constant excess of 3.5 K antenna temperature for zenithal observations, and this excess was explained by Dicke et al., 1965 (who had also constructed a radiometer, but were late to the party).

1.3 Inflation

*“To inflation! The cause of, and solution to, all of life’s problems.”*¹

The standard model of cosmology, as described above, cannot explain a few observations: The CMB has a very uniform temperature in all directions, although one would naively expect that far apart regions never had a chance to interact. Additionally, the spatial curvature we observe is close to zero, although it should have risen exponentially from a non-zero initial value and should be large even for a small perturbation.

These problems, as well as inflation as the canonically proposed solution, are discussed in more detail in the following section.

¹Adapted from Homer Simpson

1.3.1 Problems with the standard model

Horizon problem

Consider the particle horizon at scale factor a

$$w(0, a) = \int_0^a \frac{da'}{a'\dot{a}'} = \int_0^a \frac{c da'}{a'^2 H(a')} = \frac{c}{H_0} \int_0^a \frac{da'}{a'^2 E(a')} . \quad (1.71)$$

We are now interested in this horizon at recombination ($a_{\text{rec}} \simeq 10^{-3}$). The expansion function is at that time dominated by radiation and matter and takes the form

$$E(a) = \sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}} , \quad (1.72)$$

which can be further simplified by using (1.31), $a_{\text{eq}} = \Omega_{r,0}/\Omega_{m,0} \simeq 10^{-4}$, and setting $\Omega_{m,0} \simeq 1$. The comoving particle horizon is then rewritten as

$$w(0, a_{\text{rec}}) \simeq \frac{c}{H_0} \int_0^{a_{\text{rec}}} \frac{da}{\sqrt{a + a_{\text{eq}}}} = \frac{2c}{H_0} (\sqrt{a_{\text{rec}} + a_{\text{eq}}} - \sqrt{a_{\text{eq}}}) \sim 200 \text{ Mpc} . \quad (1.73)$$

This horizon at the time of recombination will correspond to a certain scale on the observed CMB sky.

To understand this scale, consider the angular distance to the recombination shell as $D_{\text{ang}}(a_{\text{rec}}) = a_{\text{rec}} f_K(w(a_{\text{rec}}, 1))$. Assuming a flat, matter-dominated Einstein-de Sitter universe (since this is a rough estimate), $E(a) = \Omega_{m,0}^{1/2} a^{-3/2}$ and the comoving distance is given by

$$w(a_{\text{rec}}, 1) \simeq \frac{c}{H_0 \sqrt{\Omega_{m,0}}} \int_{a_{\text{rec}}}^1 \frac{da}{\sqrt{a}} = \frac{2c}{H_0 \sqrt{\Omega_{m,0}}} (1 - \sqrt{a_{\text{rec}}}) . \quad (1.74)$$

To first order ($1 \gg \sqrt{a_{\text{rec}}}$) the angular distance then becomes

$$D_{\text{ang}} \simeq \frac{2ca_{\text{rec}}}{H_0 \sqrt{\Omega_{m,0}}} \simeq 15 \text{ Mpc} , \quad (1.75)$$

which is much smaller than the comoving distance due to cosmic expansion introducing a curvature to spacetime.

With this result the angular size of the (proper) particle horizon at the time of recombination becomes

$$\theta_{\text{rec}} \simeq \frac{a_{\text{rec}} w(0, a_{\text{rec}})}{D_{\text{ang}}(a_{\text{rec}})} \simeq 0.024 \simeq 1.4^\circ , \quad (1.76)$$

much smaller than the entire sky. This is problematic since we know that the CMB temperature is up to perturbations $\sim 10^{-5} \text{ K}$ very consistent in any direction, despite different regions never having had the chance to interact in this model.

Flatness problem

The curvature density parameter is defined as

$$\Omega_K = -\frac{Kc^2}{a^2 H^2} = -\frac{Kc^2}{\dot{a}^2} , \quad (1.77)$$

so its time derivative is

$$\dot{\Omega}_K = \frac{2Kc^2\ddot{a}}{\dot{a}^3} = -\Omega_K \cdot 2\frac{\ddot{a}}{\dot{a}}. \quad (1.78)$$

Assuming that $E(a) \sim a^{-n}$ (which corresponds to a type of matter with equation of state parameter w with $n = 3(1+w)/2$) one can use (1.28) to obtain $a(t) \sim t^{1/n}$. Furthermore, in this case

$$H = \dot{a}/a = (nt)^{-1} \text{ and } \ddot{a}/\dot{a} = \frac{(1/n - 1)}{t} = H(1 - n). \quad (1.79)$$

So in the end

$$\dot{\Omega}_K = \Omega_K \cdot 2H(n - 1). \quad (1.80)$$

This relation is stable for $n = 1$ or $w = -1/3$, otherwise it leads to a rapidly rising $|\Omega_K|$ if $n > 1$ or $w > -1/3$. This is the case for both dust and relativistic matter, so one would expect a large Ω_K nowadays. However, the measured value is small compared to the other density parameters and compatible with zero: $\Omega_K = 0.001 \pm 0.002$ (Planck Collaboration VI, 2020). Such a small value could be explained by fine-tuning $K = 0$ exactly, which is unsatisfying.

1.3.2 Inflation as a possible solution

Looking into the definition of Ω_K , it involves the comoving Hubble radius c/aH . It describes the size of the observable universe; if it were to shrink sufficiently for some duration in early times, the curvature density parameter could be brought very close to zero. Additionally, the observable universe could be shrunk into a causally connected region. In most models, a factor of at e^{45} to e^{60} or higher is required (Odintsov et al., 2023).

The precise requirement on a is an accelerated expansion

$$\frac{d}{dt} \left(\frac{c}{aH} \right) < 0 \Rightarrow -\frac{c\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \ddot{a} > 0, \quad (1.81)$$

which shrinks the observable region.

$\ddot{a}$ appears in Friedmann II (1.12). Without a cosmological constant the above requirement translates into $\rho c^2 + 3P < 0$, or

$$w = \frac{P}{\rho c^2} < -\frac{1}{3}. \quad (1.82)$$

A component with such an equation of state parameter straightforwardly solves the divergence issue with (1.80).

For, e.g., the cosmological constant $\dot{\rho} = 0$, so as seen in sec. 1.1.2 it follows that $w = -1$, which fulfills 1.82. This is the de Sitter case

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_{\Lambda,0}}, \quad a(t) = a_0 \exp(H_0 \sqrt{\Omega_{\Lambda,0}} t). \quad (1.83)$$

A similar solution can be reached by reformulating the condition 1.81 into

$$-\frac{\dot{H}}{H^2} < 1, \quad (1.84)$$

which prefers a slowly changing Hubble parameter. The constant case again corresponds to the de Sitter universe.

If H is not quite constant, but perturbed (e.g. $H(t) \simeq H_0 - H_0 t$) then $a(t) \sim \exp(Ht)$. This is called the quasi-de Sitter case. Such a rapid expansion fulfilling 1.81 could in principle also be achieved by power-law scale factor models, but their predicted tensor-to-scalar-ratio is larger than suggested by Planck data (Odintsov et al., 2023).

1.3.3 Canonical scalar field slow roll inflation

One very common ansatz for inflation is the so-called scalar field slow roll model. As the name suggests, one considers a self-interacting scalar field φ , dubbed the inflaton field, with a potential $V(\varphi)$. Going through the derivations, the so-called slow roll parameters ε and η turn out to be useful, limiting $\dot{\varphi}$ and the steepness of V . As usual, there is more than one way to skin a cat, with different approaches and normalizations leading to equivalent results; as for the rest of this chapter, I will be following the derivations presented by Bartelmann, 2019.

Basic ansatz

Consider the Lagrange density of a single scalar field φ with metric g

$$\mathcal{L} = -\frac{1}{2\hbar c} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) . \quad (1.85)$$

It is also possible to define the Lagrange density for multiple scalar fields $\varphi_1, \dots, \varphi_n$. Then $V(\varphi) \rightarrow V(\varphi_1, \dots, \varphi_n)$ and the first term turns into a sum over all fields. The normalization may also vary between publications (e.g. Odintsov et al., 2023, leave out the $\hbar c$).

The energy-momentum tensor is now

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L} = \frac{1}{\hbar c} \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2\hbar c} \partial_\alpha \varphi \partial^\alpha \varphi + V(\varphi) \right) . \quad (1.86)$$

Using the local limit, set $g = \eta$ to simplify the calculations. Then, one can obtain the local energy density

$$\rho c^2 = T_{00} = \frac{\dot{\varphi}^2}{2\hbar c^3} + \frac{1}{2\hbar c} (\nabla \varphi)^2 + V(\varphi) , \quad (1.87)$$

and the pressure

$$P = \frac{1}{3} T_i^i = \frac{1}{3} \left[\frac{1}{\hbar c} (\nabla \varphi)^2 - 3 \left(-\frac{\dot{\varphi}^2}{2\hbar c^3} + \frac{1}{2\hbar c} (\nabla \varphi)^2 + V(\varphi) \right) \right] \quad (1.88)$$

$$= \frac{\dot{\varphi}^2}{2\hbar c^3} - \frac{1}{6\hbar c} (\nabla \varphi)^2 - V(\varphi) . \quad (1.89)$$

Homogeneity demands that $\nabla \varphi = 0$, so the equation of state parameter is given by

$$w = \frac{\dot{\varphi}^2/2\hbar c^3 - V(\varphi)}{\dot{\varphi}^2/2\hbar c^3 + V(\varphi)} . \quad (1.90)$$

Negative pressure is fulfilled when $\dot{\varphi}^2/2\hbar c^3 < V(\varphi)$, and condition 1.82 (accelerated expansion) restricts this further to

$$\frac{\dot{\varphi}^2}{\hbar c^3} < V(\varphi) . \quad (1.91)$$

Slow roll

This condition is fulfilled by demanding slow roll (from here on $V \equiv V(\varphi)$, $V' \equiv dV/d\varphi$)

$$\dot{\varphi}^2/\hbar c^3 \ll V \quad (1.92)$$

$$2\dot{\varphi}\ddot{\varphi}/\hbar c^3 \ll V' \dot{\varphi} \quad (1.93)$$

$$2|\ddot{\varphi}|/\hbar c^3 \ll |V'|, \quad (1.94)$$

where the latter two stem from the time derivative of 1.92.

To understand the dynamics associated with such a potential, one can insert the density into Friedmann I (1.11):

$$H^2 = \frac{8\pi G}{3c^2} \left(\frac{\dot{\varphi}^2}{2\hbar c^3} + V(\varphi) \right). \quad (1.95)$$

Furthermore, one can transform the continuity equation 1.16: The time derivative of the field density is given by

$$\dot{\rho}c^2 = \frac{\dot{\varphi}\ddot{\varphi}}{\hbar c^3} + \dot{\varphi}V' \quad (1.96)$$

and noting that

$$(1+w)\rho c^2 = \rho c^2 + P = \frac{\dot{\varphi}^2}{\hbar c^3}, \quad (1.97)$$

the continuity equation takes the form

$$\ddot{\varphi} + \hbar c^3 V' = -3\dot{\varphi}H. \quad (1.98)$$

Using the slow roll conditions on (1.95) and (1.98), Friedmann I and the continuity equation turn into

$$H^2 = \frac{8\pi G}{3c^2} V \quad (1.99)$$

$$\frac{3H\dot{\varphi}}{\hbar c^3} = -V' \quad (1.100)$$

for a slow-rolling scalar field dominating the dynamics. These equations allow to rewrite the first slow roll condition 1.92 into a commonly used form

$$\frac{\dot{\varphi}^2}{\hbar c^3} = \hbar c^3 \frac{V'^2}{9H^2} = \frac{2\varepsilon}{3} V \ll V, \quad (1.101)$$

where the first slow-roll parameter ε was introduced:

$$\varepsilon := \frac{E_{\text{Pl}}^2}{2} \left(\frac{V'}{V} \right)^2 \stackrel{!}{\ll} 1, \quad E_{\text{Pl}} := \sqrt{\frac{\hbar c^5}{8\pi G}}. \quad (1.102)$$

Taking the time derivative of (1.100) gives

$$\frac{\ddot{\varphi}}{\hbar c^3} = -\frac{d}{dt} \frac{V'}{3H} = -\frac{V''\dot{\varphi}}{3H} + \frac{V'\dot{H}}{3H^2}. \quad (1.103)$$

To reformulate the last term, looking at the time derivative of (1.99), and then inserting H^2 and $\dot{\varphi}$ yields

$$2H\dot{H} = \frac{8\pi G}{3c^2} V' \dot{\varphi}, \quad \dot{H} = \frac{HV'}{2V} \dot{\varphi} = -\hbar c^3 \frac{V'^2}{6V}, \quad (1.104)$$

so

$$\frac{V' \dot{H}}{3H^2} = -\frac{E_{\text{Pl}}^2 V'^3}{6V^2} = -\frac{V'}{3} \varepsilon. \quad (1.105)$$

For the other term in (1.103), straightforward insertions give

$$-\frac{V'' \dot{\varphi}}{3H} = \frac{\hbar c^3 V'' V'}{9H^2} = \frac{E_{\text{Pl}}^2 V''}{3V} V' = \frac{\eta}{3} V', \quad (1.106)$$

where the second slow roll parameter

$$\eta := E_{\text{Pl}}^2 \frac{V''}{V} \quad (1.107)$$

was introduced.

So in total, the time derivative of the continuity equation in slow roll conditions (1.103) leads to

$$\frac{\ddot{\varphi}}{\hbar c^3} = \frac{V'}{3} (\eta - \varepsilon). \quad (1.108)$$

Using the slow roll condition (1.94), this is equivalent to

$$|\eta - \varepsilon| \ll 1 \text{ and } |\eta| \ll 1 \quad (1.109)$$

In the literature, an alternative expression of the slow roll parameters in terms of H is sometimes used. They are then written as $\epsilon_1 = -\dot{H}/H^2 = \varepsilon$, and $\epsilon_2 = \ddot{H}/2H\dot{H} = \epsilon_1 - \eta$ (e.g. Odintsov et al., 2023).

The attentive reader may have noticed that first parameter ϵ_1 embodies the accelerated expansion condition (1.84) and stands for a decreasing Hubble radius, whereas η ensures a slowly changing, flat potential. This is required for inflation to take long enough: Since $\dot{\varphi}$ must be small, the potential must also be relatively flat (as per the Euler-Lagrange equations) and its slope must change slow enough to allow for sufficient inflation.

During inflation, as a rises quickly, contributions from all other density parameters drop off steeply while the energy density of the inflaton field is held up due to its negative pressure. Inflation thus requires some mechanism to stop and convert its energy into the other components (reheating).

Inflation ending means violation of 1.81 and the end of slow roll, so the kinetic term $\sim \dot{\varphi}^2$ gains importance and depending on the shape of the potential, the field may oscillate.

Perturbations and structure formation

It is intuitive to assume that the inflaton field must have fluctuated due to Heisenberg's uncertainty principle. These early fluctuations would be pulled apart during inflation until they are larger than the horizon, essentially freezing them in place. These fluctuations would then lead to the formation of cosmic large scale structure.

For the following derivations, consider the FLRW line element written in so-called conformal time $\eta = dt/a$:

$$ds^2 = a^2(\eta)(-c^2 d\eta^2 + d\vec{w}^2) . \quad (1.110)$$

η can be calculated from t as

$$\eta = \int_0^t \frac{dt}{a} = \int_0^a \frac{da'}{a'\dot{a}'} = \frac{1}{H_0} \int_0^a \frac{da'}{a'^2 E(a')} , \quad (1.111)$$

or for constant H during inflation ($a = a_0 \exp Ht$)

$$\eta = -\frac{1}{aH} \quad \text{and} \quad a = -\frac{1}{\eta H} , \quad (1.112)$$

which is proportional to the comoving Hubble radius.

Perturbations in the metric can be generally introduced as $g_{\mu\nu}(t, \vec{x}) = g_{\mu\nu,c}(t) + \delta g_{\mu\nu}(t, \vec{x})$. The perturbations δg can be decomposed into scalar, vector, and tensor components due to symmetries in the background metric, and each component can be treated separately. It can be seen that vector perturbations are rendered unimportant during inflation and are not produced by inflation in the first place. Tensor perturbations cause gravitational waves which I will not discuss further here. Scalar perturbations cause density perturbations, which are addressed in the following derivations.

Let $\phi \ll 1$ be a field of metric perturbations in terms of a Newtonian gravitational potential so that

$$ds^2 = a^2(\eta)(-(1 + 2\phi)d\eta^2 + (1 - 2\phi)d\vec{w}^2) , \quad (1.113)$$

and, furthermore, introduce perturbations in the inflaton field

$$\varphi \rightarrow \varphi_c + \varphi , \quad (1.114)$$

where φ_c is constant in space and φ stands for the inflaton perturbations.

These perturbations are entered into the energy-momentum tensor, which is then linearized in φ and ϕ and entered into the Einstein field equations, also linearized in ϕ . Let now derivation by conformal time be denoted by $f' := df/d\eta$ and using the definitions

$$z := \frac{a^2}{a'} \varphi'_c , \quad u := a\varphi + \phi z . \quad (1.115)$$

the result of the described calculation is

$$u'' - \frac{z''}{z} u - c^2 \nabla^2 u = 0 . \quad (1.116)$$

u describes a mix of the inflaton and metric perturbation fields φ and ϕ .

Under slow roll conditions H and $\dot{\varphi}$ must be approximately constant or only very slowly changing. The derivative of the unperturbed inflaton field is then $\varphi'_c = a\dot{\varphi}_c \propto a$ and $a' = a\dot{a} = a^2 H \propto a^2$, so

$$\frac{z''}{z} = \frac{a''}{a} = \frac{1}{a} \frac{d^2}{d\eta^2} \left(-\frac{1}{\eta H} \right) = -\frac{2}{aH\eta^3} = \frac{2}{\eta^2} . \quad (1.117)$$

With this simplification, it is obvious to Fourier transform 1.116 with modes u_k :

$$u_k'' - \frac{2}{\eta^2} u_k + c^2 k^2 u = 0 . \quad (1.118)$$

This differential equation is solved by Bunch-Davies modes

$$u_k^\pm = u_k^{(0)} e^{\pm i c k \eta} \left(1 \pm \frac{i}{c k \eta} \right) , \quad (1.119)$$

where $u_k^{(0)}$ is a free prefactor.

Since u is a rather unintuitive combination of fields, it is more informative to transform back to φ and ϕ . Using $a' = a^2 H$ implies that in slow roll conditions

$$\phi z = \phi \frac{a^2}{a^2 H} \varphi'_c = \frac{\phi}{H} a \dot{\varphi}_c \ll \dot{\varphi}_c , \quad (1.120)$$

since both ϕ and $\dot{\varphi}_c$ are small. So, one can simplify $u = a\varphi$, and in turn the perturbation $\varphi = -\eta H u$. Inserting a Bunch-Davies mode u_k gives

$$\varphi_k^\pm = -\eta H u_k^\pm = \varphi_k^{(0)} \frac{H}{c k} e^{\pm i c k \eta} (c k \eta \pm i) . \quad (1.121)$$

The behavior of these solutions depends on $|c k \eta| = c k / a H$, which is the relation of the size of a mode $\sim k^{-1}$ to the comoving Hubble radius $\frac{c}{a H}$. If $|c k \eta| \ll 1$, the mode is much larger than the horizon, in the opposite case it is smaller.

Modes contained within the horizon will oscillate, and those moving out of the horizon ($|c k \eta| \ll 1$) can no longer evolve and “freeze out” at

$$\varphi_k^\pm = \pm i \varphi_k^{(0)} \frac{H_k}{c k} , \quad (1.122)$$

where H_k is the almost constant Hubble parameter when the mode crossed the horizon.

In order to calculate the exact value of $\varphi_k^{(0)}$, the field needs to be treated as a quantum field, with modes expressed using creation and annihilation operators. Assuming that the fluctuations are on the order of the Planck energy E_{Pl} one can get $\varphi_k^{(0)} = E_{\text{Pl}}^{3/2} / \sqrt{2 \hbar c k}$. The mean square amplitude of the fluctuations is then

$$\langle |\varphi_k^\pm|^2 \rangle = \frac{E_{\text{Pl}}^3}{2 \hbar} \frac{H_k^2}{c^3 k^3} . \quad (1.123)$$

Fluctuations on different scales are independent, so the total field is the sum of many independent components. Under this assumption the central limit theorem implies that the total field follows a Gaussian distribution. The search for possible non-Gaussianities is thus an important task in recent cosmology research, e.g. for distinguishing different models of inflation. I will discuss these developments in more detail in sec. 1.6.1.

From inflaton fluctuations to density fluctuations

For moving on to the generation of Large-Scale Structure (LSS), one idea is to assume that the fluctuations φ cause fluctuations δt in the time when inflation ends along the lines of

$$\delta t = \frac{\varphi}{\dot{\varphi}_c} = - \frac{3 H \varphi}{\hbar c^3 V'} , \quad (1.124)$$

where 1.100 came into play.

The energy of inflation is converted into conventional matter, and the subsequent variations in density ρ correspond to variations in the Hubble parameter according to Friedmann I, (1.11)

$$\delta\rho/\rho = \delta(H^2)/H^2 = 2\delta H/H . \quad (1.125)$$

Assuming then that the relativistic component dominates ($a \propto \sqrt{t}$), the Hubble parameter at the of reheating time is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{2t}, \quad \delta H = -\frac{\delta t}{2t^2} = -2H^2\delta t , \quad (1.126)$$

so for the density generated from the inflaton field we get

$$\frac{\delta\rho}{\rho} = -4H\delta t = -12\frac{H^2\varphi}{\hbar c^3 V'} = -\frac{4}{E_{\text{Pl}}^2} \frac{V}{V'} \varphi . \quad (1.127)$$

Entering the amplitude of a single mode $\varphi \equiv \varphi_k$ and the slow roll parameter ε (1.102) the relative density perturbations are

$$\langle |\delta_k^2| \rangle := \left\langle \left| \frac{\delta\rho_k}{\rho} \right|^2 \right\rangle = \frac{4E_{\text{Pl}}^2 H_k^2}{\hbar c^3 \varepsilon k^3} = \frac{4}{3E_{\text{Pl}}^2} \left(\frac{V}{\varepsilon k^3} \right) \Big|_{k=aH} , \quad (1.128)$$

where the right hand side again needs to be evaluated at horizon crossing.

Generally, instead of $\langle |\delta_k^2| \rangle$ publications use the scalar power spectrum

$$\mathcal{P}_s(k) := \frac{k^3}{2\pi^2} \langle |\delta_k^2| \rangle , \quad (1.129)$$

usually described by a power law with spectral index n_s , so that

$$n_s - 1 := \frac{d \ln \mathcal{P}_s(k)}{d \ln k} . \quad (1.130)$$

Here, $d \ln k$ at horizon crossing is given by

$$d \ln k = \frac{dk}{k} = \frac{d(aH)}{aH} = \frac{da}{a} = \frac{\dot{a} dt}{a} = H dt , \quad (1.131)$$

and the time interval expressed as

$$dt = \frac{d\varphi_c}{\dot{\varphi}_c} = -\frac{3H}{\hbar c^3 V'} d\varphi_c , \quad (1.132)$$

so

$$d \ln k = -\frac{3H^2}{\hbar c^3 V'} d\varphi_c = -\frac{1}{E_{\text{Pl}}^2} \frac{V}{V'} d\varphi_c . \quad (1.133)$$

In total this means

$$n_s - 1 = E_{\text{Pl}}^2 \frac{V'}{V} \frac{d}{d\varphi_c} (\ln \varepsilon - \ln V) = 2\eta - 6\varepsilon , \quad (1.134)$$

so n_s is a fundamental parameter that can be used to test the slow roll assumptions.

A similar calculation for tensorial perturbations of the metric corresponding to gravitational waves results in a tensorial power spectrum

$$\mathcal{P}_t(k) = \frac{32}{3\pi^2 E_{\text{Pl}}} V \Big|_{k=aH} , \quad (1.135)$$

with a tensorial spectral index

$$n_t := \frac{d \ln \mathcal{P}_t}{d \ln k} = -2\varepsilon . \quad (1.136)$$

An important quantity for testing models of inflation is the so-called tensor-to-scalar ratio

$$r = \frac{\mathcal{P}_t}{\mathcal{P}_s} = 16\varepsilon . \quad (1.137)$$

According to Planck Collaboration VI, 2020, the scalar spectral index and the tensor-to-scalar ratio are restricted as $n_s = 0.965 \pm 0.004$, $r < 0.06$ and they “find no evidence for deviations from a purely power-law primordial spectrum”.

Different Potentials

Odintsov et al., 2023 show calculations of the slow roll parameters and spectral indices for different potentials $V(\varphi)$.

Considering a massive scalar field

$$V = \frac{1}{2}m^2\varphi^2 , \quad (1.138)$$

the slow roll parameters can be calculated. Assuming that inflation ends when they reach $\mathcal{O}(1)$, the field value φ_{end} at the end of inflation is inferred. From there, they estimate the number of N -folds inflated during this time as

$$N = \int_{\varphi_k}^{\varphi_{\text{end}}} \frac{H}{\dot{\varphi}} d\varphi , \quad (1.139)$$

where $N \equiv \ln a_2/a_1$ measures the order of magnitude of inflation. Solving the integral under slow roll assumptions, one can get the value of the field at horizon crossing depending on a given N . Reinserting this value into the slow roll parameters allows for the estimation of the scalar spectral index and the tensor-to-scalar ratio as

$$n_s \simeq \frac{6N-3}{6N+1} , \quad r \simeq \frac{16}{6N+1} . \quad (1.140)$$

For $N = 60$, this gives $n_s \simeq 0.9889$, which is incompatible with observations.

As an alternative, they discuss the self-interacting scalar field

$$V(\varphi) = \lambda\varphi^4 . \quad (1.141)$$

The same considerations as above yield

$$n_s \simeq \frac{N-2}{N+1} , \quad r \simeq \frac{16}{N+1} . \quad (1.142)$$

Again, $n_s \simeq 0.95$ and $r \simeq 0.26$, which is again incompatible with Planck data.

An example for a non-power law potential is the so-called natural inflation model with potential

$$V(\varphi) = L^4(1 \pm \cos(\varphi/f)) , \quad (1.143)$$

where $L \sim M_{\text{GUT}}$ and $f \sim M_{\text{Pl}}$ are scales for the potential height and width². Taylor expanding for small φ/f gives power law potentials as described above. Inflation takes place as the field moves towards the potential minimum at πf . The slow roll parameters depend on f only and one gets

$$\sin(\varphi_k/f) \simeq \sin(\varphi_{\text{end}}/f) \exp\left(-\frac{NM_{\text{Pl}}^2}{16\pi f^2}\right) . \quad (1.144)$$

This relation sets an upper limit on the value of $\varphi_k \lesssim 0.1 M_{\text{Pl}}$ at the start of inflation.

Generally, the Planck collaboration find good agreement between data and single field slow roll models (Planck Collaboration XXII, 2014; Planck Collaboration XX, 2016; Planck Collaboration X, 2020). The allowed potential shapes, however, are restricted to plateau-like potentials (Ijjas et al., 2013). This leads to problems with both tight initial conditions for starting inflation and a smaller range of allowed parameters requiring fine-tuning, and large regions of simple potential shapes leading to disagreement with the data.

1.3.4 Alternative models of inflation

Canonical minimally-coupled scalar field slow roll inflation as derived in the previous subsection leaves several open questions – especially the nature of the inflaton field and the corresponding potential, as well as mechanisms for reheating are not explained further. There exist a multitude of alternative models using various approaches, and it is impossible to discuss all of them here. The interested reader is pointed to more specialized reviews, e.g. Odintsov et al., 2023; Nojiri et al., 2017; Martin et al., 2024. I will only provide a very general overview to some of them.

Alternative models are expected to produce tensor-to-scalar ratios and spectral indices complying with observations. In some of them, those parameters depends on the scale k , in which case one refers to a running spectral index. The tensor spectral index is challenging to derive from observations due to the weak nature of B-modes (curl) in the CMB polarization field. More models can be distinguished using the primordial gravitational wave spectrum created from tensor perturbation modes. NANOGrav results (Agazie et al., 2023) may point to the existence of such a background, but a different cosmological origin cannot be fully excluded at the time of writing.

More complicated models of inflation include non-minimally coupled models. Such models may contain multiple fields, or curvature corrections in terms of the Ricci scalar R , called $f(R)$ gravity corrected canonical scalar field models. These models also introduce more slow roll parameters (more on this in e.g. Nojiri et al., 2017).

Another set of alternative models is the constant roll evolution. In slow roll, the kinetic term is overshadowed by the potential (1.92). Following from this condition and the continuity equation, the (absolute) acceleration $\ddot{\varphi}$ of the field is negligible compared to the potential slope (1.94). In constant roll, however, it is fixed according to

$$\ddot{\varphi} = \beta H \dot{\varphi} , \quad (1.145)$$

² M_{GUT} refers to the mass scale predicted by Grand Unified Theories that corresponds to the energy where the electromagnetic, weak, and strong forces unite

where β is a dimensionless, not necessarily constant parameter. It is possible, but not necessary, to use $\beta \ll 1$ as a slow roll condition. Now, the second order derivative of φ in the continuity equation (1.98) remains relevant. Constant roll can introduce primordial scalar non-Gaussianities.

There are also models introducing string correction terms, leading to a more complicated action depending on more parameters.

1.4 Large Scale Structure

The perturbations of the inflaton field provide seeds for the Large-Scale Structure (LSS) observable today. The variations in gravitational potential lead to gravitational collapse of the surrounding matter, which will be described in this section.

The structures visible today (galaxy clusters, filaments, and larger) extend on scales $\lesssim 50$ Mpc, which is significantly smaller than the Hubble radius (~ 4 Gpc). Thus the following description takes place in the Newtonian regime.

Conservation of mass is contained in the continuity equation

$$\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0 . \quad (1.146)$$

Additionally, the Euler equation (conservation of momentum) with gravitational potential Φ is given by

$$\dot{\vec{v}} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{\rho} - \nabla \Phi , \quad (1.147)$$

with Φ following the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho . \quad (1.148)$$

These equations describe an ideal fluid with negligible mean free path and a well-defined velocity field. If, e.g., it occurred that two streams of matter were to cross without mixing, this approximation would break down. The relations above follow from the energy-momentum tensor $T^{\mu\nu}$ through

$$\partial_\mu T^{\mu\nu} = 0 . \quad (1.149)$$

In the relativistic case, the partial derivative needs to be replaced by a covariant derivative.

Converting everything into comoving coordinates, the velocity separates into

$$\vec{v} = \frac{d}{dt} \vec{r} = \dot{a} \vec{x} + a \dot{\vec{x}} =: \vec{v}_0 + a \vec{u} , \quad (1.150)$$

where $\vec{v}_0 = \dot{a} \vec{x}$ is the Hubble velocity and $\vec{u} := \dot{\vec{x}}$ is the peculiar velocity. With this definition, the three relations

$$\vec{v}_0 \cdot \nabla = a H \vec{x} \cdot \nabla , \quad \nabla \cdot \vec{v}_0 = 3\dot{a} , \quad \text{and} \quad (\vec{y} \cdot \nabla) \vec{v}_0 = \dot{a} \vec{y} \quad (1.151)$$

hold.

The gradient transforms into the comoving gradient ∇_x as

$$\nabla = a^{-1} \nabla_x . \quad (1.152)$$

The partial time derivative at constant physical coordinates $\vec{r}$ differs from the one at constant comoving coordinates $\vec{x}$. This can be seen by comparing $df(t, \vec{r})$ and $df(t, \vec{x})$, which results in

$$\partial_t \rightarrow \partial_t - H\vec{x} \cdot \nabla , \quad (1.153)$$

where the gradient is taken with respect to comoving coordinates.

Inserting both gradient and time derivative for comoving coordinates into the continuity equation 1.146 gives

$$\dot{\rho} - H\vec{x} \cdot \nabla \rho + a^{-1} \nabla(\rho \vec{v}) = 0 , \quad (1.154)$$

the Euler equation 1.147 turns into

$$\dot{\vec{v}} - H(\vec{x} \cdot \nabla) \vec{v} + a^{-1} (\vec{v} \cdot \nabla) \vec{v} = - \frac{\nabla P}{a\rho} - a^{-1} \nabla \Phi . \quad (1.155)$$

and the same is done for the Poisson equation 1.148

$$\nabla^2 \Phi = 4a^2 \pi G \rho . \quad (1.156)$$

Let the density contrast δ be defined so that

$$\rho = \rho_0(1 + \delta) . \quad (1.157)$$

The continuity and Euler equations must also hold in the unperturbed case (no variation in density, pressure, and gravitational potential) as

$$\dot{\rho}_0 + a^{-1} \nabla(\rho_0 \vec{v}_0) = 0 \quad (1.158)$$

and

$$\dot{\vec{v}}_0 - H(\vec{x} \cdot \nabla) \vec{v}_0 + a^{-1} (\vec{v}_0 \cdot \nabla) \vec{v}_0 = 0 . \quad (1.159)$$

With density contrast and potential perturbation ϕ , the perturbed Poisson equation can be written as

$$\nabla^2(\Phi_0 + \phi) = 4\pi G a^2 \rho_0(1 + \delta) . \quad (1.160)$$

At this point, one can pretend that the background density ρ_0 causes the potential Φ_0 and the density variations $\rho_0 \delta$ cause ϕ . This is a bad assumption in the Newtonian case, because then, for a constant Φ_0 the density $\rho_0 = 0$. However, a more thorough assessment using general relativity gives the Newtonian expectation for the perturbations

$$\nabla^2 \phi = 4\pi G a^2 \rho_0 \delta . \quad (1.161)$$

Pressure and density perturbations are connected via the speed of sound

$$\delta P = \left(\frac{\partial P}{\partial \rho} \right) \delta \rho = c_s^2 \rho_0 \delta . \quad (1.162)$$

Combining the above gives the perturbed, but not linearized, continuity and Euler equations

$$\dot{\delta} + \nabla \cdot ((1 + \delta) \vec{u}) = 0 \quad (1.163)$$

and

$$\dot{\vec{u}} + 2H\vec{u} + (\vec{u} \cdot \nabla) \vec{u} = - \frac{c_s^2 \nabla \delta}{a^2(1 + \delta)} - \frac{\nabla \phi}{a^2} . \quad (1.164)$$

In linearized form, these simplify to

$$\dot{\delta} + \nabla \cdot \vec{u} = 0 \quad (1.165)$$

$$\dot{\vec{u}} + 2H\vec{u} = -\frac{c_s^2 \nabla \delta + \nabla \phi}{a^2} \quad (1.166)$$

Taking the time derivative of 1.165 and the divergence of 1.166 to eliminate $\nabla \cdot \vec{u}$ gives a differential equation in only δ

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_0\delta + \frac{c_s^2 \nabla^2 \delta}{a^2} \quad (1.167)$$

for non-relativistic matter.

Repeating the same derivations for relativistic matter, it turns out that

$$\ddot{\delta} + 2H\dot{\delta} = \frac{32\pi G}{3}\rho_0\delta + \frac{c^2 \nabla^2 \delta}{3a^2} . \quad (1.168)$$

These differential equations are linear and homogeneous, so the solutions are additive, and one can Fourier transform them into

$$\ddot{\delta}_k + 2H\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right) \delta_k \quad (1.169)$$

$$\ddot{\delta}_k + 2H\dot{\delta}_k = \left(32\pi G\rho_0 - \frac{c^2 k^2}{a^2}\right) \delta_k/3 \quad (1.170)$$

for the non-relativistic and relativistic case, respectively, and with spatial modes $\delta_k(t)$.

In the static (undamped) case of $H = 0$, both turn into simple harmonic oscillators with frequency

$$\omega_0^2 = \begin{cases} \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\right) & \text{non-relativistic} \\ \left(\frac{c^2 k^2}{a^2} - 32\pi G\rho_0\right)/3 & \text{relativistic} . \end{cases} \quad (1.171)$$

When the right hand side is positive, the frequency takes a real value and the perturbations oscillate. The corresponding length scale is the comoving Jeans length defined as

$$\lambda_J := \frac{2\pi}{k_J} = \begin{cases} \frac{c_s}{a} \sqrt{\frac{\pi}{G\rho_0}} & \text{non-relativistic} \\ \frac{c}{2a} \sqrt{\frac{\pi}{2G\rho_0}} & \text{relativistic} \end{cases} \quad (1.172)$$

For a proper treatment of the different dynamics of relativistic and non-relativistic matter, the time dependency should be replaced by the scale factor using

$$\frac{d}{dt} = \dot{a} \frac{d}{da} = aH \frac{d}{da} , \quad \frac{d^2}{dt^2} = (aH)^2 \frac{d^2}{da^2} + (H + aH')aH \frac{d}{da} . \quad (1.173)$$

The density is then rephrased in terms of the critical density $\rho_0 = \frac{3H^2}{8\pi G}\Omega$, so that equations 1.169 and 1.170 become (E = expansion function)

$$\delta_k'' + \left(\frac{3}{a} + \frac{E'}{E}\right) \delta_k' = \left(\frac{3\Omega_m}{2} - \frac{c_s^2 k^2}{a^2 H^2}\right) \frac{\delta_k}{a^2} \quad (1.174)$$

$$\delta_k'' + \left(\frac{3}{a} + \frac{E'}{E}\right) \delta_k' = \left(4\Omega_r - \frac{c^2 k^2}{3a^2 H^2}\right) \frac{\delta_k}{a^2} . \quad (1.175)$$

In the radiation dominated era $E \simeq 1/a^2$ and $E'/E = -2/a$. Large scale modes of relativistic matter ($k \ll 2\sqrt{3}aH/c$) are solved by a polynomial $\delta_k \sim a^{\pm 2}$, where only the non-vanishing solution will be relevant. On scales significantly smaller than the horizon, the modes oscillate. Non-relativistic matter during that era uses $\Omega_m = 0$, so on large scales or for pressure free matter ($c_s = 0$) the whole right-hand side of the equation is zero. The solution consists of a constant term plus $\ln a$, making these modes grow only very slowly. Smaller modes or those with non-negligible speed of sound will oscillate.

In the matter dominated era $E \sim a^{-3/2}$ and $E'/E = -3/2a$. For pressure free, non-relativistic matter the solution is given by a power law either growing with a or falling with $a^{-3/2}$, independent of k . Again, the growing solution will dominate.

These growing modes lead to the formation of voids and overdensities. In a universe with non-negligible cosmological constant and matter terms, the density is generally described by $\delta(a) = \delta_0 D_+(a)$ with the linear growth factor $D_+(a)$ approximated by

$$D_+(a) = \frac{5a}{2} \frac{\Omega_m}{\Omega_m^{4/7} - \Omega_\Lambda + (1 + \Omega_m/2)(1 + \Omega_\Lambda/70)} . \quad (1.176)$$

Structures are growing slower and slower due to the accelerated expansion, so the structures observable today must have generated earlier.

The oscillation length scale, below which modes oscillate instead of growing is defined by the speed of sound. However, for collisionless dark matter that is ill-defined. Then, the Jeans length must be calculated from the velocity dispersion σ_v using

$$\lambda_J = \frac{1}{a} \langle \sigma_v^{-2} \rangle^{-1/2} \sqrt{\frac{\pi}{G\rho_0}} \quad (1.177)$$

Here, instead of the medium's pressure and sound waves, the particle's individual motion counteracts gravity at small scales. In the so-called Cold Dark Matter scenario $\sigma_v = 0$ and its Jeans length is negligible, so growth occurs at all scales. This is not the case for warm or hot dark matter.

The model using cold dark matter (CDM) in combination with a cosmological constant ($w = -1$) driving late-time expansion is called Λ CDM. It is often treated as a default model other assumptions are tested against.

Perturbations in velocity

Consider again the linearized Euler equation 1.166. Neglecting the density gradient, that is, assuming the velocity field is only generated from potential perturbations with no previous movement, the velocity is parallel to the potential gradient:

$$\dot{\vec{u}} + 2H\vec{u} = -\frac{\nabla\phi}{a^2} . \quad (1.178)$$

The straightforward ansatz $\vec{u} = -u(t)\nabla\phi$ is now inserted into the continuity equation 1.165, which gives the relation

$$\dot{a} \frac{d\delta}{da} = \dot{\delta} = -\nabla\vec{u} = u(t)\nabla^2\phi = u(t) 4\pi G a^2 \rho_0 \delta , \quad (1.179)$$

where in the last step the Poisson equation 1.161 was used. For linear perturbations

$$\frac{d\delta}{da} = \frac{\delta}{a} \frac{d \ln D_+(a)}{d \ln a} = \frac{\delta}{a} f(\Omega_m) , \quad (1.180)$$

and setting $f(\Omega_m) \simeq \Omega_m^{0.6}$ happens to be a very good approximation. Additionally, replacing

$$4\pi G \rho_0 = 4\pi G \frac{3H^2}{8\pi G} \Omega_m = \frac{3}{2} H^2 \Omega_m \quad (1.181)$$

gives for the velocity field

$$u(t) = \frac{2f(\Omega_m)}{3a^2 H \Omega_m} , \quad \delta\vec{v} = a\vec{u} = -\frac{2f(\Omega_m)}{3a H \Omega_m} \nabla\phi . \quad (1.182)$$

This expression describes matter falling into a potential valley, and the valley will deepen in turn. There also exist rotating solutions, where $\vec{u} = \nabla \times \vec{p}$ for some vector potential p . The continuity equation for such solutions gives a constant density contrast, so such modes cannot be generated under the physics assumed here.

1.4.1 Power spectrum

There is no relation assumed between the different Fourier modes of the density field. Modes smaller than the horizon oscillate and those larger freeze out. The total field is thus composed of a sum of many independent components of equal origin and the central limit theorem holds; the field is Gaussian.

Gaussian fields are fully defined by their mean and variance. The density contrast has zero mean by definition. The variance as a function of k is called the power spectrum and formally defined as

$$(2\pi)^3 \delta_D(\vec{k} - \vec{k}') P_\delta(k) := \langle \delta_k(t) \delta_{k'}^*(t) \rangle , \quad (1.183)$$

with the Dirac delta distribution δ_D due to homogeneity and isotropy, and the asterisk denotes the complex conjugate. The density contrast in location space is a real field, so $\delta_k^*(t) = \delta_{-k}(t)$.

In an expanding universe the Hubble radius and, hence, the length scale of causally connected perturbations is growing. So, in the radiation dominated phase, relativistic modes grow $\sim a^2$ until they enter the horizon (or rather, the horizon grows to contain them), and start oscillating. The distribution of non-relativistic matter is influenced by these modes that are able to grow larger the later they enter the horizon. Later, in the matter dominated phase, all modes of matter with negligible pressure and velocity dispersion grow linearly $\sim a$. Modes entering the horizon before matter dominance are thus intermittently blocked from growing.

For the mode k_{eq} entering at radiation-matter equality a_{eq} (with a comoving Hubble radius of $\sim 93 \text{ Mpc}$) we have

$$a_{\text{eq}} \lambda_{\text{eq}} = a_{\text{eq}} \frac{2\pi}{k_{\text{eq}}} \stackrel{!}{=} r_{\text{H,eq}} = \frac{c}{H_0} \frac{a_{\text{eq}}^{3/2}}{\sqrt{2\Omega_{m,0}}} , \quad (1.184)$$

so the wave number entering the horizon just then is given by

$$k_{\text{eq}} = 2\pi \frac{H_0}{c} \sqrt{\frac{2\Omega_{m,0}}{a_{\text{eq}}}} . \quad (1.185)$$

Larger modes with smaller wave numbers keep growing throughout; smaller modes stop growing upon entering the horizon at a_{enter} .

In the radiation dominated era, the Hubble radius is proportional to a^2 , and afterwards to $a^{3/2}$, so for a_{enter} it must hold, that

$$a_{\text{enter}} \lambda = a_{\text{enter}} \frac{2\pi}{k} \stackrel{!}{=} r_H \propto a_{\text{enter}}^2 \text{ for small scale modes, else } r_H \propto a_{\text{enter}}^{3/2} . \quad (1.186)$$

Thus, $a_{\text{enter}} \propto k^{-1}$ ($a_{\text{enter}} < a_{\text{eq}}$) and $a_{\text{enter}} \propto k^{-2}$ ($a_{\text{enter}} > a_{\text{eq}}$)

The power spectrum is affected by this evolution. For modes entering the horizon late $a_{\text{enter}} \propto k^{-2}$, so $P_{\text{enter}}(k) = a_{\text{enter}}^{-2} P_i(k) = k^{-4} P_i(k)$. Modes entering during radiation dominance have $a_{\text{enter}} \propto k^{-1}$, but $P_{\text{enter}}(k) = a_{\text{enter}}^{-4} P_i(k)$, so here as well $P_{\text{enter}}(k) = k^{-4} P_i(k)$.

In the section on inflation, the power spectrum of inflationary perturbations at horizon crossing was given in 1.129 as

$$\mathcal{P}_s(k) = \frac{k^3}{2\pi^2} \langle |\delta_k|^2 \rangle$$

and it could be written as a power law $\propto k^{n_s-1}$. So the spectrum of density perturbations upon entering the horizon is given by

$$P_{\text{enter}}(k) \propto \langle |\delta_k|^2 \rangle \propto k^{n_s-4} \quad (1.187)$$

and at early times

$$P_i(k) \propto k^{n_s} . \quad (1.188)$$

If n_s were equal to one, the initial spectrum would be scale invariant and the variance of density perturbations would be independent of the time of horizon entry. But since inflation cannot have taken arbitrarily long, the scale invariance is imperfect.

As seen above, for $k < k_{\text{eq}}$ all modes grow equally, but modes with larger k that enter the horizon before a_{eq} are suppressed by a factor $\sim f_{\text{sup}}^2$ with the suppression factor

$$f_{\text{sup}} = \left(\frac{a_{\text{enter}}}{a_{\text{eq}}} \right)^2 = \left(\frac{k_{\text{eq}}}{k} \right)^2 . \quad (1.189)$$

In total, for ideal CDM one expects

$$P(k) \propto \begin{cases} k & k < k_{\text{eq}} \\ k^{-3} & k \gg k_{\text{eq}} \end{cases} \quad (1.190)$$

The maximum of the power spectrum at around k_{eq} now allows for the calculation of the current matter density $\Omega_{m,0}$.

For hot dark matter with a significant velocity dispersion hindering clustering, the power spectrum will be falling exponentially above the Jeans wave number.

High density contrast regime

The conclusions in this section so far relied on the perturbations to be small with respect to the mean density. An approach to high-contrast dynamics still in the linear regime is the Zel'dovich approximation (Zel'dovich, 1970).

Consider particles following trajectories generally described by

$$\vec{r}(t) = a(t)\vec{x} + b(t)\vec{f}(\vec{x}) , \quad (1.191)$$

where $\vec{x}$ is an initial position of the particle at early times and the displacement field $\vec{f}$ is assumed to be rotation free (a possible divergence free component would vanish over the course of expansion). Then, $\vec{f}$ can be written as the divergence of a potential ψ

$$\vec{f}(\vec{x}) = \nabla\psi(\vec{x}) . \quad (1.192)$$

Particle dynamics are then governed by the time coordinate $b(t)$.

The number of trajectories is conserved, so the density is given by the Jacobi determinant of $\vec{x} \mapsto \vec{r}$,

$$\rho = \rho_0 \det^{-1} \left[\frac{\partial r_i}{\partial x_j} \right] . \quad (1.193)$$

Introduce now the deformation tensor $f_{ij} := \partial f_i / \partial x_j = \partial^2 \psi / \partial x_i \partial x_j$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. The density can then be shown to be given by

$$\rho = \frac{\rho_0}{(a + b\lambda_1)(a + b\lambda_2)(a + b\lambda_3)} , \quad (1.194)$$

where ρ_0 is the initial mean density. It evolves geometrically in the expanding space with $\bar{\rho} = \rho_0 a^{-3}$, so the density contrast

$$\delta = \frac{1}{(1 + b/a \lambda_1)(1 + b/a \lambda_2)(1 + b/a \lambda_3)} - 1 \simeq -\frac{b}{a}(\lambda_1 + \lambda_2 + \lambda_3) = -\frac{b}{a} \nabla \vec{f} . \quad (1.195)$$

One can see, that

$$\frac{b}{a} \equiv D_+(a) \text{ and } \delta_0 \equiv -\nabla \vec{f} . \quad (1.196)$$

Furthermore, δ must fulfill the continuity equation 1.165, yielding $\nabla \vec{u} = -\dot{\delta}$ and giving

$$-\dot{\delta} = \left(\frac{\dot{b}}{a} - \frac{\dot{a}b}{a^2} \right) \nabla \vec{f} = H \left(\frac{db(a)}{da} - \frac{b}{a} \right) \nabla \vec{f} \stackrel{!}{=} \nabla \vec{u} , \quad (1.197)$$

so the velocity field is given by

$$\vec{u} = H \left(\frac{db(a)}{da} - \frac{b}{a} \right) \vec{f} \quad (1.198)$$

with

$$\frac{db}{da} = D_+ + a \frac{dD_+}{da} = D_+(1 + f(\Omega)) \quad (1.199)$$

and hence

$$\vec{u} = HD_+(a)f(\Omega)\nabla\psi . \quad (1.200)$$

As a result, the trajectory is described by

$$\vec{r} = a \left(\vec{x} + \frac{\vec{u}}{Hf(\Omega)} \right) . \quad (1.201)$$

Further analyzing the probability distribution of the eigenvalues of f_{ij} , it can be shown that two eigenvalues cannot be equal (Doroshkevich, 1970). Collapse of cosmic matter takes place anisotropically into pancake-like structures. At their intersections, filaments and knots form.

Non-linear evolution

As soon as regions with very high density form, even the Zel'dovich approximation breaks down as particle paths may cross. Gaussianity breaks down as different Fourier modes are no longer uncorrelated. The evolution in this regime is usually tackled using numerical approaches of various kinds. One of the earliest simulations was performed by Davis et al., 1985, who used 23 768 particles. Modern simulations may follow $\mathcal{O}(10^9)$ particles and can include hydrodynamics and smaller scale feedback processes (e.g. Dolag et al., 2016).

1.4.2 Observations of the matter power spectrum

The power spectrum is closely related to the two point correlation function ξ via Fourier transform

$$\xi(t, y) := \langle \delta(t, \vec{x}) \delta(t, \vec{x} + \vec{y}) \rangle \quad (1.202)$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \delta_k(t) \delta_{k'}^*(t) \rangle \exp(i\vec{k}\vec{x}) \exp(i\vec{k}'(\vec{x} + \vec{y})) \quad (1.203)$$

$$= \int \frac{d^3k}{(2\pi)^3} P(k) \exp(-i\vec{k}\vec{y}) . \quad (1.204)$$

The variance σ of the density field is then

$$\sigma = \xi(0) = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) . \quad (1.205)$$

The value of σ depends on the scale at which the density contrast is measured. Usually, the field is smoothed with a convolution

$$\bar{\delta} := \int d^3y \delta(t, \vec{x}) W_R(|\vec{x} - \vec{y}|) , \quad (1.206)$$

where W_R is a smoothing kernel with scale R . In Fourier space, a convolution corresponds to a multiplication of the modes, so as a function of the power spectrum, the variance is given by

$$\sigma_r^2 = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) W_{kR}^2 . \quad (1.207)$$

Publications will usually give the value of σ_8 at $R_8 := 8/h \text{ Mpc}$ as a measure for clustering³. Criticism of this practice has been raised in recent years, since this radius is dependent on h , and some suggest the use of $\sigma_{12}/R_{12} = 12 \text{ Mpc}$ as a Hubble-independent scale (Sánchez, 2020; Forconi et al., 2025).

³Remember h is defined so that $H_0 = h \cdot 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

Observational biases

Measuring the correlation function in practice is challenging due to the generally low completeness of galaxy (cluster) surveys and the uncertainties in positions; especially since uncertainties in the correlation function at large scales highly affect the small scale power spectrum.

One can also attempt to infer the power spectrum directly from the distribution of N galaxies, using a single delta distribution for each object. This ansatz adds a plane wave to each mode in Fourier space. A single mode has the variance

$$\langle \delta_k \delta_k^* \rangle = \frac{V^2}{N^2} \sum_{j=1}^N e^{-i\vec{k}\vec{x}_j} e^{i\vec{k}\vec{x}_j} = \frac{V^2}{N} , \quad (1.208)$$

where V is the survey volume. This term is a shot noise that needs to be subtracted from the estimated power spectrum.

More problems arise due to masking and irregularly shaped survey volumes. Not the actual density is observed, but rather its product with a window function f

$$1 + \delta(\vec{x}) \rightarrow f(\vec{x})(1 + \delta(\vec{x})) . \quad (1.209)$$

This multiplication turns into a convolution when taking the Fourier transform, which smoothens the apparent power spectrum and lowers the amplitude. The latter can be corrected with

$$P_{\text{obs}}(k) \rightarrow P_{\text{obs}}(k) \frac{(\int_V f d^3x)^2}{V \int_V f^2 d^3x} . \quad (1.210)$$

Additionally, galaxies are usually weighted according to their local number density, so that galaxies from densely populated regions do not overwhelm the overall distribution, and the survey volume may not be suitable to estimate the true mean number density.

The considerations above were valid for the number density power spectrum. Moving on to the density contrast, the so-called bias factor is introduced for each mode

$$\delta n_k = b(k) \delta_k , \quad (1.211)$$

and its effect on the power spectrum

$$P^{\text{gal}}(k) = b^2(k) P(k) . \quad (1.212)$$

The bias factor encodes that, e.g., galaxy clusters need to be weighted more heavily than single galaxies. It can be estimated from secondary effects, such as redshift space distortions of gravitational lensing.

The 3D coordinates of galaxies are usually obtained by using their redshift as a measure of distance, but galaxies will also have a peculiar velocity as they move with respect to each other, leading to the so-called redshift space distortion

$$D = \frac{v_{\text{Hubble}} + v_{\text{pec}}}{H_0} = D_{\text{true}} + \delta D . \quad (1.213)$$

Galaxies are mapped to a spot $\vec{s}$ with

$$\vec{s} = \vec{x} + \frac{\vec{u}}{H} , \quad (1.214)$$

where $\vec{x}$ is the true location, H needs to be the local Hubble function near the cluster. Using u as the peculiar velocity projected onto the line of sight, the density contrast is shifted by

$$\delta \rightarrow \delta - \frac{1}{H} \frac{du}{dx} . \quad (1.215)$$

This can be derived using the Jacobi determinant of $\partial \vec{s} / \partial \vec{x}$.

The peculiar velocity $\vec{u}$ is also given by 1.182 and the potential gradient. Relating that term to the density via the Poisson equation, and projecting the corresponding real space velocity onto the line of sight, a single mode of the derivative with respect to x is given by

$$\frac{du}{dx} = \int \frac{d^3 k}{(2\pi)^3} i k_\mu u_k e^{i k x \mu} \quad (1.216)$$

$$\left(\frac{du}{dx} \right)_k = i k_\mu u_k = -H f(\Omega_m) \mu^2 \delta_k , \quad (1.217)$$

where μ is the cosine of the angle between $\vec{k}$ and the line of sight.

So in total, the density contrast power spectrum is modified as

$$P(k) \rightarrow P_s(k) = P(k) \left(1 + f(\Omega_m) \mu^2 \right)^2 . \quad (1.218)$$

Adding the bias factor from the number count

$$\delta n_k \rightarrow b \delta_k + f(\Omega_m) \mu^2 \delta_k = \delta n_k (1 + \beta \mu^2) \quad (1.219)$$

with $\beta := \frac{f(\Omega_m)}{b}$. So in total, the redshift space density contrast power spectrum is distorted by a quadrupole

$$P_s^{\text{gal}}(k) = P^{\text{gal}}(k) (1 + \beta \mu^2)^2 . \quad (1.220)$$

Galaxy clusters also show the finger-of-god effect, where high peculiar velocities make the cluster appear stretched in redshift space.

Baryonic Acoustic Oscillations (BAO) before recombination left an imprint on the matter distribution on scales related to the sound horizon derived in the next section (1.237). The corresponding k is of the same order of magnitude as k_{eq} , so the power spectrum is expected to oscillate when going to scales beyond its maximum. This effect is visible in measurements and can be used as a distance measure when combined with theoretical expectations.

A notable recent survey of BAO was performed at the Dark Energy Spectroscopic Instrument (DESI), whose DR2 cosmology results show signs favoring evolving dark energy (DESI Collaboration et al., 2025).

Concerning n_s , recent measurements show a spectral power law index close to, but significantly below one, e.g. Planck Collaboration VI, 2020 find $n_s = 0.965 \pm 0.004$, and recent measurements by Liu et al., 2025 agree with $n_s = 0.966^{+0.019}_{-0.015}$, in agreement with Λ CDM. It should be noted that recent data from the James Webb Space Telescope show an unexpectedly high density of massive galaxies at redshift $z > 10$, which remains unexplained by Λ CDM (Menci et al., 2022).

1.5 Structures in the CMB

The early perturbations discussed in the previous sections are not only visible in the Large-Scale Structure of galaxies, but also as anisotropies in the CMB. Again, a few secondary effects are at play here.

The kinematic dipole

An observer attached to the Earth or the Solar System is not in the CMB's rest frame, so a very prominent kinematic red-/blueshift along the relative movement is generated. This shift does not affect the blackbody shape of the spectrum. The shift in temperature depends on the direction $\vec{n}$ as

$$T_{\text{obs}} = \frac{\omega_{\text{obs}}}{\omega_{\text{CMB}}} T_{\text{CMB}} = \frac{1}{\gamma(1 - \vec{v}_{\text{CMB}} \cdot \vec{n}/c)} T_{\text{CMB}} \quad (1.221)$$

Expanding this to second order in v_{CMB}/c , and introducing the angle α between line of sight and CMB,

$$T_{\text{obs}} \simeq T_{\text{CMB}} \left(1 + \cos(\alpha) \frac{v_{\text{CMB}}}{c} + (\cos^2(\alpha) - 1/2) \frac{v_{\text{CMB}}^2}{c^2} \right). \quad (1.222)$$

The first order term corresponds to a dipole modulation and the second order term to a quadrupole, which is significantly weaker than the dipole. This kinematic dipole is usually subtracted from the CMB maps.

The Planck collaboration have found the speed $v = (369.82 \pm 0.11) \text{ km s}^{-1}$ corresponding to $3362.08 \pm 0.99 \mu\text{K}$ with movement in the direction $(l, b) = (264.021 \pm 0.011, 48.253 \pm 0.005)$ (Planck Collaboration III, 2020).

Magnitude of CMB anisotropies

The matter density contrast observed today is substantially larger than one on smaller scales. Going back to the time of recombination using the growth function, it must have been

$$\delta(a_{\text{CMB}}) = \delta^{(0)} D_+(a_{\text{CMB}}) \gtrsim a_{\text{CMB}} \simeq 10^{-3}. \quad (1.223)$$

If the energy density contrast imprinted in the CMB were the same, we should observe variance on the order of millikelvins. The fluctuations visible however are $\mathcal{O}(10^{-5}) \text{ K}$. This discrepancy can be solved by the assumption that cold dark matter dominates structure growth, and is able to collapse before recombination since it is unaffected by the photon bath, giving it a higher density contrast.

For a more accurate account of the CMB fluctuations, one would need to start from the Boltzmann equation for the photon distribution in phase space and include relativistic effects such as spacetime curvature and gravitational signal delay. I will again in this section follow the simplified description of Bartelmann, 2019.

From here on, assume only a pure photon gas, without cosmic expansion, and ignoring the difference between physical and comoving coordinates. As for matter, perturbations in pressure, density, and velocity are introduced and the corresponding equations linearized. The unperturbed equations are then subtracted from the perturbed equations to get equations for the perturbations only.

The perturbed continuity and Euler equations end up as

$$\rho_0 \dot{\delta} + \left(\rho_0 + \frac{P_0}{c^2} \right) \nabla \vec{v} = 0 \quad (1.224)$$

$$\dot{\vec{v}} = -\nabla \phi - \frac{c^2 \nabla \delta P}{\rho_0 c^2 + P_0} . \quad (1.225)$$

In a photon gas, $w = 1/3$,

$$P = \frac{\rho c^2}{3} , \quad (1.226)$$

and the energy density is given by $u = \rho c^2$, which scales with T^4 along with the pressure, so

$$u_0 + P_0 = 4P_0 , \quad \frac{\delta P}{P_0} = 4 \frac{\delta T}{T_0} . \quad (1.227)$$

With the introduction of the temperature variation

$$\Theta := \frac{\delta T}{T_0} , \quad (1.228)$$

equations 1.224 and 1.225 turn into

$$\dot{\Theta} + \frac{1}{3} \nabla \vec{v} = 0 \quad (1.229)$$

$$\dot{\vec{v}} = -\nabla \phi - c^2 \nabla \Theta . \quad (1.230)$$

Combining those by eliminating $1/3 \nabla \vec{v}$ yields a single equation for Θ ,

$$\ddot{\Theta} - \frac{c^2}{3} \nabla^2 \Theta = -\square \Theta = \frac{1}{3} \nabla^2 \phi . \quad (1.231)$$

Fourier transforming again leads to an equation for temperature perturbation modes

$$\ddot{\Theta}_k + \frac{c^2 k^2}{3} \Theta_k = -\frac{k^2}{3} \phi_k . \quad (1.232)$$

This is a wave equation for temperature perturbations fueled by gravity potential perturbations, that are in turn driven by density perturbations. The waves are sound waves of speed $c/\sqrt{3}$, as the Laplacian of Θ stems from the pressure gradient in 1.225. This acoustic behavior can be used to define a sound horizon at the time of recombination.

The photons are coupled to a baryonic gas, which increases the inertia to be taken into account; the baryonic pressure is negligible. The mean photon energy at recombination is much smaller than the electrons' rest energy, $0.3 \text{ keV} \ll 511 \text{ keV}$. Scattering then takes place in the non-relativistic regime and is described by Thomson scattering.

The increased inertia changes the Euler equation, here described by introducing a factor

$$\vec{v} \rightarrow (1 + R) \vec{v} , \quad \nabla \phi \rightarrow (1 + R) \nabla \phi , \quad (1.233)$$

with the time dependent ratio of mass densities

$$R := \frac{\rho_B c^2}{u_0 + P_0} = \frac{3a \Omega_{B,0}}{4 \Omega_{\gamma,0}} . \quad (1.234)$$

The Euler equation 1.225 becomes

$$(1 + R)\dot{\vec{v}} + \dot{R}\vec{v} = -(1 + R)\nabla\phi - c^2\nabla\Theta . \quad (1.235)$$

Dividing by $3(1 + R)$, taking the divergence, and subtracting the result from the time derivative of the continuity equation eliminates $\nabla\vec{v}$ and gives the modified wave equation

$$\ddot{\Theta} - \frac{c^2}{3(1 + R)}\nabla^2\Theta + \frac{\dot{R}\dot{\Theta}}{1 + R} = \frac{1}{3}\nabla^2\phi . \quad (1.236)$$

Now the speed of sound is decreased as $c/\sqrt{3(1+R)}$, and the term $\sim \dot{\Theta}$ introduces dampening due to the baryons.

The comoving sound horizon for this speed of sound is given by

$$r_s(a) = \frac{c}{H_0} \int_0^a \frac{da'}{a'^2 E(a') \sqrt{3(1 + R)}} . \quad (1.237)$$

Acoustic oscillations can only form for density perturbations smaller than this limit.

The primordial Sachs-Wolfe effect

A relativistic effect influencing the observable temperature variations is the so-called primordial Sachs-Wolfe effect (Sachs and Wolfe, 1967). Perturbations of the metric expressed through the Newtonian gravitational potential ϕ cause a varying redshift $\delta\omega$ for photons that contributes to the temperature perturbations

$$\frac{\delta\omega}{\omega_0} = \frac{\phi}{c^2} = \frac{\delta T}{T_0} . \quad (1.238)$$

The potential perturbations also cause the photons to be released under a time delay by $\delta t/t_0 = -\phi/c^2$ in which scale factor changes $a \propto t^{2/3}$

$$\frac{\delta a}{a_0} = \frac{2}{3} \frac{\delta t}{t_0} = -\frac{2}{3} \frac{\phi}{c^2} , \quad (1.239)$$

so to an observer these photons will be compared to photons emitted at $T_0 \sim a^{-1}$ at a later time

$$\frac{\delta T_0}{T_0} = \frac{2\phi}{3c^2} . \quad (1.240)$$

In total, the temperature shift is then

$$\frac{\delta T}{T_0} = \Theta = \frac{\phi}{3c^2} \quad (1.241)$$

Temperature variations are therefore created not only through sound waves or density perturbations, but also through potential perturbations. This is the primordial Sachs-Wolfe effect; there is also the integrated Sachs-Wolfe effect, which is a foreground effect on CMB photons moving through changing potential wells on their way to the observer.

Sound waves follow the CDM power spectrum proportional to $\sim k^{n_s}$ up to k_{eq} , and above that $\sim k^{n_s-4}$. The gravity potential however follows the Poisson equation with $-k^2\phi_k \propto \delta_k$, so its power spectrum

$$P_\delta \propto k^4 P_\phi , \quad (1.242)$$

which is proportional to k^{-3} for larger and k^{-7} for smaller scales. It is heavily suppressed compared to the acoustic perturbations at small scales, and only relevant at the largest scales.

Silk dampening

At small scales a different secondary effect comes into play as the mean free path of photons gets relevant, especially near end of recombination as the cosmic plasma clears up (Silk, 1968).

A photon on random walk with N steps over a mean free path of $(n_e \sigma_T)^{-1}$ will travel an average distance of

$$\lambda_D = \sqrt{N} (n_e \sigma_T)^{-1} , \quad (1.243)$$

with the Thomson cross section σ_T . The amount of interactions per time interval is $dN = n_e \sigma_T c dt$, so the mean square diffusion length is given by

$$d\lambda_D^2 = \frac{cdt}{n_e \sigma_T} \quad (1.244)$$

Structures smaller than this will get smoothed for wave numbers $k \gtrsim \frac{2\pi}{\lambda_D}$ due to photon diffusion. This smoothing is called Silk dampening.

Polarization

Thomson scattering is sensitive to polarization, and if the light falling into a region is anisotropic, so will be the overall emission of that region. Electrons on the last scattering surface get illuminated by anisotropic light, so the CMB is somewhat linearly polarized with the polarized part having an amplitude $\sim 10\%$ of the intensity anisotropies.

1.5.1 Classical CMB analysis

We observe the CMB projected onto the celestial sphere. In order to get from 3D perturbations to such a visible spherical map, different modes are mixed in the projection to spherical harmonics Y_l^m ,

$$\exp(i\vec{k}\vec{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l j_l(kr) Y_l^m(\vec{e}_r) Y_l^{m*}(\vec{e}_k) , \quad (1.245)$$

with the spherical Bessel functions $j_l(kr)$ smearing the modes.

Here, the decomposition of a field Θ in spherical harmonics with complex coefficients a_{lm} is used

$$\Theta(\vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\vartheta, \varphi) . \quad (1.246)$$

The Y_l^m form an orthogonal and complete set, so the coefficients a_{lm} can be calculated by

$$a_{lm} = \int_0^{2\pi} d\varphi \int_0^\pi \sin \vartheta d\vartheta \Theta(\vartheta, \varphi) Y_l^m(\vartheta, \varphi) . \quad (1.247)$$

The power spectrum is then given by

$$C_l := \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \sum_{m=-l}^l |a_{lm}|^2 . \quad (1.248)$$

Often, publications instead use a differently normalized form giving the total power contained in each mode,

$$\mathcal{D}_l := \frac{l(l+1)}{2\pi} C_l . \quad (1.249)$$

The power spectrum as measured by Planck is shown in Fig. 1.1. One can see that on large scales greater than the sound horizon the Sachs-Wolfe effect dominates, followed by acoustic perturbations, and on small scales Silk dampening sets in. The exact shape is sensitive to cosmological parameters.

1.5.2 Foreground effects

CMB photons traverse the observable universe and as such are subject to a multitude of foreground effects.

Point-source foregrounds

The photons interact with both gravitational potentials and the matter in galaxy clusters, leading to secondary effects observable in the CMB.

Trivially, highly redshifted galaxies can directly impact CMB observations with their infrared emission.

An important further contribution comes from the Sunyaev-Zeldovich (SZ) effect (Sunyaev and Zeldovich, 1972). In its thermal version, photons traversing a hot gas get upscattered by inverse Compton scattering with high energy electrons. As a result, the photon spectrum is deformed in a way quantified by the Compton parameter, which depends on the thermal photon energy, the electron density and the Thomson cross section. Galaxy clusters will darken the CMB below ~ 217 GHz and brighten it above that, and disappear as sources at exactly that frequency. There is also the kinetic SZ effect caused by electrons moving along with galaxy clusters.

As a counterpart to the primordial Sachs-Wolfe effect, there is the integrated Sachs-Wolfe effect. As photons move through a changing gravitational potential, their energy is shifted, allowing to infer changes in cosmic structures on the way to the observer.

Gravitational potentials will also bend photon paths along their way. This introduces an astigmatism to the CMB field washing out small scale perturbations. It can be roughly described by an index of refraction calculated from Newtonian potential variations

$$n = 1 - \frac{2\phi}{c^2} . \quad (1.250)$$

This lensing leads to the coupling of different modes, from which its angular scales and magnitude can be estimated. It gains relevance at scales $\lesssim 10'$.

Additionally, solar system objects can emit in the submillimeter range, which can be used to detect dwarf planets with CMB probes.

Diffuse foregrounds

Diffuse foreground emission largely originates in the Milky Way.

Synchrotron radiation stems from relativistic electrons in magnetic fields. It is visible around the Galactic Plane but also from filaments outside and highly polarized. It follows a power law spectrum falling from radio to microwave ranges.

Dust in the Milky Way can have temperatures around 10-20 K, which is warmer than the CMB and therefore the spectrum is rising in the CMB range. Part of the emission is self-absorbed by the dust. It also has a polarized component, dubbed spinning dust.

HII regions emit Bremsstrahlung with an exponentially falling spectrum, and they are concentrated on the Galactic Disc. There is also a contribution of Bremsstrahlung from molecular clouds.

Locally, zodiacal light from within the solar system may disturb observations. It is caused by sunlight scattering with dust along the ecliptic plane.

In total, observations of the CMB are best performed between 100 and 200 GHz, and many frequency bands are required to separate the foregrounds' differing spectral shapes from the CMB.

1.5.3 Observations

The CMB spectrum has its maximum around

$$\nu_{\max} = 160.18 \text{ GHz}, \quad \lambda_{\max} = 1.87 \text{ mm} . \quad (1.251)$$

Silk dampening suppresses structures below scales of a few arcminutes, so for a supposed required resolution of five arcminutes, a telescope diameter of

$$D = 1.22 \frac{\lambda}{\Delta\theta} \simeq 157 \text{ cm} \quad (1.252)$$

is needed. At this size, photon shot noise would remain relevant for shorter surveys of, e.g., one year.

In order to calculate the power spectrum, for each l there are $2l + 1$ modes with different values of m , so only that many measurements can be effectively taken. By symmetry, only half of those modes are independent, resulting in the cosmic variance

$$\sigma_l^2 = \frac{2C_l^2}{2l + 1} . \quad (1.253)$$

This is a fundamental limit stemming from the nature of the power spectrum and cannot be improved with better instruments.

Observations in the CMB frequency range need to be done either outside of the earth's atmosphere or in very dry environments, such as the Atacama desert or the South Pole due to absorption by water vapor. The earliest space-based observations of the CMB were performed by the Cosmic Background Explorer (COBE) satellite from 1989 to 1993, which detected evidence for Gaussian temperature anisotropies (Smoot et al., 1992). These measurements were vastly improved by the Wilkinson Microwave Anisotropy Probe (WMAP), which enabled the determination of cosmic parameters with unprecedented precision (Bennett et al., 2003). It was in operation from 2001 to 2010.

Notable modern ground-based CMB observatories include the South Pole Telescope (SPT, see e.g. Sobrin et al., 2022, for the latest generation) and the Atacama Cosmology Telescope (ACT, see e.g. Madhavacheril et al., 2024; Mallaby-Kay et al., 2021). Due to their limited field of observation, they are restricted to the analysis of the power spectrum at higher modes.

Here, however, I will use data from the Planck mission as the most recent space-based CMB data set.

Planck

Planck started its survey observations in 2009. It performed five all-sky surveys with two instruments over 30 months, and a few additional months with only its Low Frequency Instrument (LFI) after depleting the coolant required for the High Frequency Instrument (HFI). It took observations in 9 channels between 30 and 857 GHz. These frequencies required very cool detectors (HFI had to be kept below 100 mK). These 9 channels allow to separate foregrounds from the original signal. The Planck Collaboration has released maps of temperature and polarization over the course of several data releases, publicly available on the Planck Legacy Archive. The CMB temperature maps are of a high enough quality that all expected interesting features are depicted, enabling subsequent projects to focus on polarization alone. A scientific overview is given in Planck Collaboration I, 2020.

The most recent maps contain a small residual dipole, since the previous 2015 data release subtracted the solar dipole instead of the best fit dipole (the kinematic quadrupole is also subtracted). Maps are available for four different component separation approaches: Commander, NILC, SEVEM, and SMICA. They are described in the component separation paper Planck Collaboration IX, 2016.

The Planck temperature power spectrum is of very good quality across all orders physically relevant and limited by cosmic variance (Fig. 1.1, top). One can see the acoustic peaks in the center, the Silk dampened tail at high wave numbers, and at low wave numbers the primordial Sachs-Wolfe influence and high cosmic variance. The data fits very well with Λ CDM.

1.6 Anomalies and tensions

1.6.1 Anisotropy and non-Gaussianity

Although generally wide agreement between the CMB data and the standard model of cosmology, especially statistical isotropy, is observed, possible deviations have been found. These are e.g. reviewed in papers by the Planck Collaboration (see e.g. Planck Collaboration VII, 2020, for the most recent data release) and partially summarized here.

At larger scales ($l \lesssim 70$) anisotropic features are observed that are consistent between Planck and WMAP temperature data and thus most likely not of systematic origin. The fundamental issue here lies in distinguishing outliers produced by a random Gaussian field from physical deviations from isotropy or Gaussianity, and residual foregrounds. Issues may arise from not properly accounting for the “look-elsewhere-effect”, where the choice of statistics used depends on the outliers found in the data, with regions or statistical measures standing out in the data being compared to the average of simulations.

One example is the lack of large-angle correlations (Copi et al., 2015; Bennett et al., 2003) where the temperature data appears to have comparably low 2-point-correlation values between 60 and 180 deg. The Planck Collaboration VII, 2020 find this effect to be heavily dependent on the quadrupole and specific parameter choices; upon more careful consideration of outliers in the simulated maps and polarization data they claim that the anomaly is likely a statistical fluke.

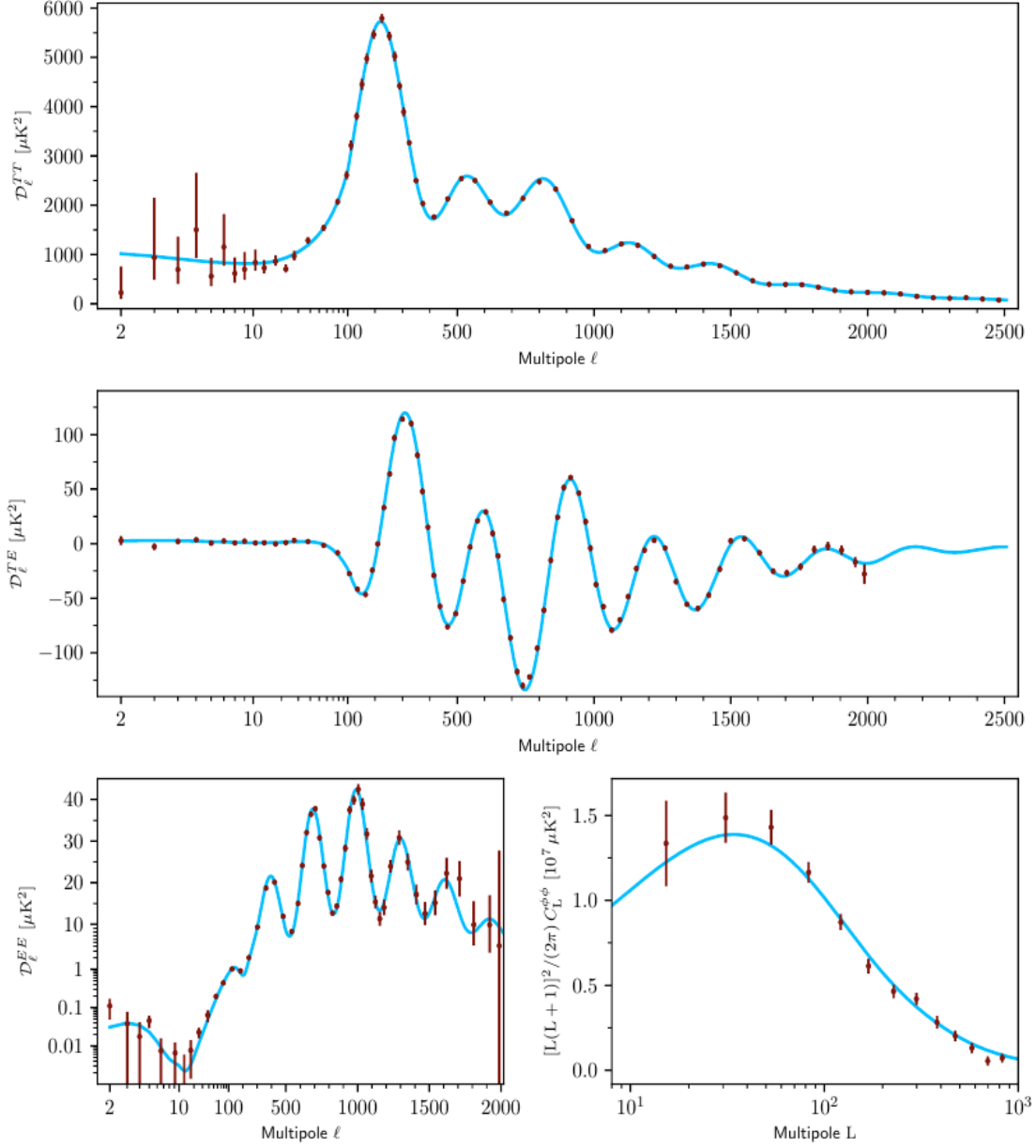


Figure 1.1: “Planck CMB power spectra. These are foreground-subtracted, frequency-averaged, cross-half-mission angular power spectra for temperature (top), the temperature-polarization cross-spectrum (middle), the E mode of polarization (bottom left), and the lensing potential (bottom right).” (Planck Collaboration I, 2020, Fig. 9). Blue lines show the best-fitting ΛCDM model.

Similarly, they argue against a possible asymmetry of N-point correlation functions when calculated for single hemispheres (Eriksen et al., 2005; Planck Collaboration XXIII, 2014; Planck Collaboration VII, 2020), finding a dependence of p -values on the component-separation pipeline (pointing towards foregrounds as a possible cause) and arguing that the significance is overestimated due to testing a large number of configurations and a posteriori parameter choices.

Furthermore, some studies have found hints towards a point-parity asymmetry where odd modes appear stronger than even modes (Land and Magueijo, 2005; Kim and Naselsky, 2010; Gruppuso et al., 2011; Planck Collaboration XVI, 2016). Planck Collaboration VII, 2020 find that this effect is reduced when using the 2018 common mask.

There also exist several large-scale peaks in the temperature distribution (see e.g. Planck Collaboration VII, 2020, Fig. 29). A prominent example is the so-called Cold Spot around $(l, b) = (210, -57)$ that was first discovered in WMAP data (Vielva et al., 2004; Cruz et al., 2005; Vielva, 2010) and later confirmed in Planck data (Planck Collaboration XXIII, 2014; Planck Collaboration XVI, 2016).

On the LSS side, inhomogeneities in the galaxy cluster X-ray luminosity-temperature relations have been found (Migkas and Reiprich, 2018; Migkas et al., 2020). The cluster brightness in X-rays is closely related to the temperature of the intracluster medium via a scaling relation (Kaiser, 1986), and can thus be used to infer luminosity distances. An anisotropy has been confirmed in various different scaling relations and cluster catalogs also in other wavelengths (Migkas et al., 2021). This phenomenon may be explained by, e.g., bulk flows, which would need to be larger than predicted by Λ CDM, or spatial variations in H_0 .

Alternative explanations

An early work of Tolman, 1934b considered inhomogeneous dust models and advised caution in interpreting too much into local homogeneity as primordial inhomogeneities may not be ironed out (that was however, before the proposal of inflation). Nowadays, a multitude of possible origins have been suggested that may create anisotropies.

One class of generally anisotropic, homogeneous cosmological models are the Bianchi models (see e.g. Ellis, 2006, for a review). They are characterized according to their 3-dimensional symmetry groups, and not all of them allow for an inflationary era. There would be an observable impact on LSS, nucleosynthesis, and the CMB, but no signs were found in the data (Planck Collaboration XVIII, 2016; Planck Collaboration XXVI, 2014).

Another proposed origin of anisotropies are cosmic strings (Kibble, 1976), which are 1D topological defects stemming from primordial symmetry breaking. They are predicted to cause temperature discontinuities in the CMB (e.g. Planck Collaboration XXV, 2014), and their properties are constrained by gravitational wave data (EPTA Collaboration et al., 2024; Abbott et al., 2019a; Abbott et al., 2019b).

The global cosmic topology might also be nontrivial, generating correlations in the CMB field (e.g. Akrami et al., 2024).

Non-Gaussianities

Apart from these directional anomalies, there has been an ongoing search for departures from Gaussian statistics in the CMB field that might distinguish alternative inflation

models (see, e.g., Bartolo et al., 2004; Komatsu, 2010, for pre-Planck reviews). Even in the standard inflation scenario, non-Gaussianities are expected from higher order perturbations on scales of $\mathcal{O}(\varepsilon, \eta)$ (Bartolo et al., 2004).

Since there is no general way of quantifying such deviations, any method condensing non-Gaussianity into a few numbers by construction can only cover specific types. One common family of parameters is f_{NL} , which is calculated via the bispectrum (or the three-point-correlation function). To this end, one chooses a triangle shape for the calculation of the angular bispectrum (such as equilateral or orthogonal, see e.g. Komatsu, 2010, for further details) and then finds an estimator for the expected probability density function of the temperature field; for a purely Gaussian field the bispectrum vanishes. In various models, a non-zero f_{NL} is predicted based on inflaton field dynamics, e.g. quantified by the slow roll parameters (see also Odintsov et al., 2023).

Planck Collaboration IX, 2020 find for different configurations $f_{\text{NL}}^{\text{local}} = -0.9 \pm 5.1$, $f_{\text{NL}}^{\text{equil}} = -26 \pm 47$, and $f_{\text{NL}}^{\text{ortho}} = -38 \pm 24$, where the errors give 68% confidence limits. One sees that the errors are large with respect to the various f_{NL} , not placing hard limits on models.

Planck Collaboration VII, 2020 also tested for signs of non-Gaussianities using different methods (N -point correlations, Minkowski functionals, Peak counts, Peak profiles), finding no evidence for departures from ΛCDM .

1.6.2 Further problems in comparison to other datasets

There are issues that arise when comparing CMB based parameter estimates to those obtained via other means. The most prominent ones recently are the Hubble tension and the growth tension, both summarized here.

Hubble tension

The Hubble tension refers to a discrepancy between values of H_0 obtained from CMB experiments (using the derived sound horizon) and those obtained via distance-ladder based measurements of the local universe. While not strictly relevant to this thesis, the problem shall be summarized here as it is a central issue in recent cosmology literature. For a summary, the reader may also be referred to sec. 5.4 of Planck Collaboration VI, 2020.

As stated before, Planck Collaboration VI, 2020 find, using the CMB, $H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}$, which is in tension with more local measurements. The issue first came up in comparison to a measurement by Riess et al., 2011, who got $H_0 = (73.8 \pm 2.5) \text{ km s}^{-1} \text{ Mpc}^{-1}$ using Cepheid variables as distance indicators as part of the “Supernovae and H_0 for the Equation of State” (SH0ES) program. The problem has worsened since then, with error bars shrinking and mean values almost constant; e.g. Riess et al., 2019 find $H_0 = (74.03 \pm 1.42) \text{ km s}^{-1} \text{ Mpc}^{-1}$. Data from other CMB experiments such as ACT are in agreement with the Planck value; Louis et al., 2025 find $H_0 = (68.22 \pm 0.36) \text{ km s}^{-1} \text{ Mpc}^{-1}$, their data being consistent with ΛCDM (Calabrese et al., 2025). BAO data gives $H_0 = (68.5 \pm 0.58) \text{ km s}^{-1} \text{ Mpc}^{-1}$, but is at tension with the CMB-derived value for $\Omega_{m,0}$ (DESI Collaboration et al., 2025). There are numerous suggestions on alleviating this tension beyond modifying the cosmological model, such

as issues with the distance ladder (Perivolaropoulos, 2024). The issue remains far from resolved.

Growth tension

Another tension has come up concerning the growth of LSS as measured by $f(z)\sigma_8$, where f is the derivative of the linear growth rate and the discrepancy occurs in σ_8 (1.180). Measurements via redshift space distortions lead to significantly lower values than via the CMB (Macauley et al., 2013), especially at lower redshift (e.g. Qu et al., 2024, and references therein). The use of σ_8 has recently been criticized as coupling the Hubble and growth tensions due to the dependence of the length scale $R_8 = 8h^{-1}$ Mpc on H_0 (Sánchez, 2020; Forconi et al., 2025).

Extended models

There are several prominent models derived in an attempt to alleviate the tensions above, also referenced in the sources in the previous paragraphs. Since this is a detour widely out of scope of this thesis, I will keep this section very short and point the interested reader to the sources in the given references.

One avenue are modifications to the neutrino sector in terms of neutrino masses and the number of effective neutrino flavors (e.g. Bennett et al., 2020). Since neutrinos are typically relativistic particles, they affect the early cosmic evolution before their decoupling. The latter also holds for early dark energy models, which introduce a dark energy term in early times (e.g. Gómez-Valent et al., 2021). Late-time changes are introduced by dynamical dark energy models with a changing equation of state parameter, a popular one of which is the w_0w_a CDM parametrization. w_0w_a here stands for a scale dependent dark energy equation of state parameter in first order given by $w(a) = w_0 + w_a(1 - a)$ (Linder, 2003). This affects late time structure formation and is currently favored by DESI results (DESI Collaboration et al., 2025).

Chapter 2

Minkowski Functionals and Tensors

The physical analysis of spatial structures requires powerful quantitative tools from geometry. While a plethora of such tools is widely known, such as N -point functions and harmonic decompositions, I will describe and apply here lesser known morphological characteristics, known as Minkowski functionals and tensors (Chiu et al., 2013). They contain information undetected by other shape measures, such as wavelets (Räth et al., 2009b) or the mean intercept length (Klatt et al., 2017b). Their advantages consist of intuitive interpretation, calculation in real space directly on image data, and complete description of convex shapes. Additionally, masked data can be handled straightforwardly. I will further explain these points in the subsequent chapter. The analysis of structures furthermore requires the analyst to be well nourished. Along with this section I recommend a fine french cheese, such as a petit vinageois.

Large parts of the content of this chapter have already been published in similar form in Collischon et al., 2024. The segments and figures original to that paper are marked with disclaimers and references.

2.1 Minkowski functionals

Minkowski functionals (MF) (also known as Quermaßintegrale) are powerful scalar shape descriptors for continuous, motion-invariant, and additive information. Their applications include statistical physics (Mecke, 1998; Mecke and Stoyan, 2000), trabecular bone structure (Räth et al., 2008; Klatt et al., 2017b; Callens et al., 2021), biomedicine of the eye (Barbosa et al., 2014; Barbosa et al., 2019), nuclear pasta matter (Schuetrumpf et al., 2015), porous media (Arns et al., 2003; Klatt et al., 2017a; Armstrong et al., 2019), structures in noisy images (Klatt and Mecke, 2020; Göring et al., 2013), and cosmological data (see chapters 3 and 4).

The MF form a basis of functionals of convex shapes with these properties as proven by Hadwiger, 1957. In d -dimensional space there exist $d+1$ MF, in 2D (up to prefactors) given by area, perimeter, and Euler characteristic (the number of connected bodies minus the number of holes).

They can then be written for a body K with local contour curvature κ as

$$W_0 = \int_K dr, \quad W_1 = \int_{\partial K} dr, \quad W_2 = \int_{\partial K} \kappa(r) dr. \quad (2.1)$$

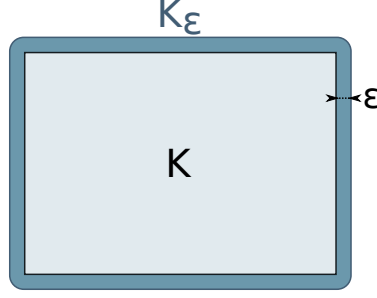


Figure 2.1: The construction of the parallel body K_ε (bounded by the outer blue line) from K (black boundary).

Different publications may use different constant prefactors; for the sake of simplicity I will omit those here as for all following chapters only relative and not absolute values are of relevance.

A fundamental way to define the MF in $\mathbb{R}^d$ is via the Steiner formula. Let K be a convex body and K_ε ($\varepsilon > 0$) be the set of all points closer than ε to K . K_ε is called the parallel body (see Fig. 2.1). Its volume is given by a polynomial in ε whose coefficients can be used to define the MF as

$$V(K_\varepsilon) = \sum_{\nu} \binom{d}{\nu} \varepsilon^\nu W_\nu(K) . \quad (2.2)$$

They are additive up to overlap

$$W_\nu(A \cup B) = W_\nu(A) + W_\nu(B) - W_\nu(A \cap B) , \quad (2.3)$$

making them easily applicable to pixel images, where the body is defined by individual pixels. Note that the MF are not generally continuous for non-convex bodies where counterexamples can be constructed (Schröder-Turk et al., 2010). These counterexamples are, however, of no concern for the applications described here. For Gaussian fields, the Gaussian kinematic formula gives ensemble expectation values (Adler and Taylor, 2007) and for mildly non-Gaussian fields approximations have been derived (Gay et al., 2012; Fantaye et al., 2015; Matsubara and Kuriki, 2021). A thorough review of their use in image processing including examples is given by Michielsen and De Raedt, 2001.

They can also be straightforwardly adapted to the surface of the sphere, where an analogous theorem to Hadwiger holds as proven by Klain and Rota, 1997. In this way they have been used, e.g., for CMB analysis (see sec. 3.1).

2.2 Minkowski tensors

MF are limited to rotation and translation invariant information due to their scalar nature. Hence, their descriptive power is limited when the relative alignment of structures or the symmetries of a shape are of interest – two galaxies of very different shape may have the same area, perimeter and Euler characteristic. For a more comprehensive shape description including symmetries and preferred directions, tensorial extensions have been developed. These are the so-called Minkowski tensors (MT). An early overview of

their development and use is given by Beisbart et al., 2002, and their mathematical background is reviewed by Hug et al., 2008.

They have been used, e.g., in statistical physics (Klatt et al., 2022), for the detection of bubble-like structures in astronomical images (Collischon et al., 2021), characterizing structures and fluid demixing in complex plasmas (Böbel and R  th, 2016; B  bel et al., 2018), quantifying the shape of galaxies (Rahman and Shandarin, 2004), or gamma ray astronomy data analysis (G  ring, 2012). Their use in CMB analysis is summarized in sec. 3.1.

2.2.1 Cartesian MT

A straightforward way to define the MT in 2D uses the position $\vec{r}$ and surface normal vectors $\vec{n}$:

$$W_0^{a,b} = \int_K \vec{r}^a \otimes \vec{n}^b dr, \quad W_\nu^{a,b} = \int_{\partial K} \vec{r}^a \otimes \vec{n}^b G_\nu dr, \quad (2.4)$$

where $\nu \in \{0, 1\}$, $G_0 = 1$ and $G_1 = \kappa$, and $\vec{r}^a \otimes \vec{n}^b$ is the symmetric tensor product given in components by

$$(\vec{r}^a \otimes \vec{n}^b)_{i_1 \dots i_{a+b}} = \frac{1}{(a+b)!} \sum_{\sigma \in S_{a+b}} r_{i_{\sigma(1)}} \dots r_{i_{\sigma(a)}} \cdot n_{i_{\sigma(a+1)}} \dots n_{i_{\sigma(a+b)}}. \quad (2.5)$$

Here, S_n is the permutation group of n components.

For example, consider $a = 1$ and $b = 2$. Then

$$\begin{aligned} (\vec{r}^1 \otimes \vec{n}^2)_{i_1 i_2 i_3} &= \frac{1}{6} \sum_{\sigma \in S_3} r_{i_{\sigma(1)}} \cdot n_{i_{\sigma(2)}} n_{i_{\sigma(3)}} \\ &= \frac{1}{6} (r_{i_1} n_{i_2} n_{i_3} + r_{i_1} n_{i_3} n_{i_2} + r_{i_2} n_{i_1} n_{i_3} \\ &\quad + r_{i_2} n_{i_3} n_{i_1} + r_{i_3} n_{i_1} n_{i_2} + r_{i_3} n_{i_2} n_{i_1}). \end{aligned}$$

Here, one can especially note the symmetry in all indices.

The MT introduce explicit position-dependence via $\vec{r}$ while orientation dependence is included in the moments of $\vec{n}$. Alesker's theorem states that any motion covariant, convex-continuous and additive mapping can be written as a linear combination of MT multiplied with the d -dimensional unit matrix (Alesker, 1999). There are, however, linear dependencies between different MT (Schneider and Weil, 2008).

Note that the MF can be written as rank zero MT:

$$W_\nu = W_\nu^{0,0} \quad (2.6)$$

As a result, I will henceforth use the abbreviation MT only, while still referring to MF and the higher-order tensors.

Some of the MT have a direct physical meaning, e.g. $W_0^{1,0}/W_0 = 1/v \int_K \vec{r} dr$ is the center of mass for an object of uniform density, and in three dimensions $-W_0^{2,0} + \text{tr}(W_0^{2,0}) \mathbf{1}_3$ is the moment of inertia (Schr  der-Turk et al., 2011).

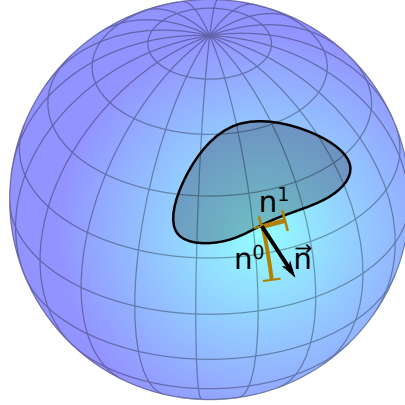


Figure 2.2: A schematic depiction of a normal vector $\vec{n}$ (black) and its components (orange) for a shape (gray) on the sphere (blue). From Collischon et al., 2024, Fig. 1.

2.2.1.1 Adaptation to the sphere

The following descriptions in this subsection have already been published in Collischon et al., 2024.

On the sphere no suitable analog for the position-dependent tensors containing $\vec{r}$ was found. I therefore restrict the description to the translation invariant $W_\nu^{0,b}$.

Instead of embedding the shape (including $\vec{n}$) into 3D, I remain in S_2 , treating $\vec{n}$ as tangent space vectors with elements $n^0 = n^\theta$ and $n^1 = n^\phi$ using the common spherical coordinate system (see Fig. 2.2). The $\vec{n}$ are normalized with the corresponding spherical metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}. \quad (2.7)$$

Then, $\vec{n}$ can be parallel transported along curves according to the rules of differential geometry.

The elements of the rank b MT can then be written as $W^{i_1 \dots i_b}$, where the indices are inherited from the symmetric tensor product (eq. 2.5). In general, these tensors contain information on b -fold symmetry, as will be shown in the following sections.

2.2.1.2 Scalar representations

In order to access anisotropy information from the Cartesian MT, they need to be brought into a meaningful scalar form. Usually one wants measures for the degree and direction of the anisotropy, however there is no known general technique that achieves such a decomposition for all ranks. Only ranks 1, 2, and 4 have so far been tackled; $W_\nu^{0,1}$ being only non-zero on curved spaces or for non-closed curves.

Rank 2 has been discussed in several publications (e.g. Chingangbam et al., 2017; Joby et al., 2019) due to its straightforward representations and comprehensive interpretation. A thorough discussion in the Euclidean plane is presented by Schröder-Turk et al., 2010; Schröder-Turk et al., 2011. I will use the nomenclature used in Collischon et al., 2024 for continuity with the previous sections, but different approaches are equivalent. The

MT can be regarded as a 2×2 matrix. Taking the trace of $W_1^{0,2}$ is equal to the contour length:

$$\text{tr} \left(W_1^{0,2}(K) \right) = \int_{\partial K} (\vec{n} \otimes \vec{n})^{\mu\nu} g_{\nu\mu} dr = \int_{\partial K} (n^0)^2 + \sin^2(\theta)(n^1)^2 dr = \int_{\partial K} dr = W_1^{0,0}(K) \quad (2.8)$$

Additionally, one can calculate its eigenvalues and eigenvectors to gain information on preferred directions and the extent of two-fold anisotropy. To this end, one may set up the eigenvalue equation for an eigenvalue λ and vector v^α as

$$W^{\mu\nu} g_{\mu\alpha} v^\alpha = \lambda v^\alpha, \quad (2.9)$$

which can be written in matrix form as

$$\begin{pmatrix} W^{00} & W^{01} \sin^2(\theta) \\ W^{10} & W^{11} \sin^2(\theta) \end{pmatrix} \vec{v} = \lambda \vec{v}. \quad (2.10)$$

Note that $W_1^{0,2}$ is symmetric, so its eigenvectors are orthogonal. The ratio of eigenvalues can be used as a measure for two-fold symmetry — if a shape is elongated in one direction, one eigenvector will point in that direction and the corresponding eigenvalue will be larger than the other. Furthermore, when analyzing a collection of several shapes (e.g. elongated grains) the eigenvalues will give information on their relative alignment. Collischon et al., 2024 introduced the term eigenvalue quotient (EVQ) for the larger divided by the smaller eigenvalue, and other publications may use the inverse definition.

Rank 4 can be approached via its complete symmetry. I do this derivation along the lines of Mehrabadi and Cowin, 1990, who wrote three-dimensional rank 4 tensors as 6×6 matrices that can be approached with the known and common methods. The approach shown here for 2D rank 4 tensors on the sphere has been published in Collischon et al., 2024.

Tensor multiplication of a symmetric rank 4 tensor W with a symmetric rank 2 tensor σ can be written as

$$W^{\alpha\beta\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \sigma^{\mu\nu} = \sigma'^{\alpha\beta} \quad (2.11)$$

or in components

$$W^{0000} g_{00} g_{00} \sigma^{00} + 2 \cdot W^{0001} g_{00} g_{11} \sigma^{01} + W^{0011} g_{11} g_{11} \sigma^{11} = \sigma'^{0,0} \quad (2.12)$$

$$W^{0100} g_{00} g_{00} \sigma^{00} + 2 \cdot W^{0101} g_{00} g_{11} \sigma^{01} + W^{0111} g_{11} g_{11} \sigma^{11} = \sigma'^{0,1} = \sigma'^{1,0} \quad (2.13)$$

$$W^{1100} g_{00} g_{00} \sigma^{00} + 2 \cdot W^{1101} g_{00} g_{11} \sigma^{01} + W^{1111} g_{11} g_{11} \sigma^{11} = \sigma'^{1,1}, \quad (2.14)$$

where the factors of two stem from the symmetry of all tensors involved. This process can be represented by a 3×3 matrix multiplied with a 3D vector:

$$\begin{pmatrix} W^{0000} & 2 \cdot W^{0001} \sin^2(\theta) & W^{0011} \sin^4(\theta) \\ W^{0100} & 2 \cdot W^{0101} \sin^2(\theta) & W^{0111} \sin^4(\theta) \\ W^{1100} & 2 \cdot W^{1101} \sin^2(\theta) & W^{1111} \sin^4(\theta) \end{pmatrix} \begin{pmatrix} \sigma^{00} \\ \sigma^{01} \\ \sigma^{11} \end{pmatrix} = \begin{pmatrix} \sigma'^{00} \\ \sigma'^{01} \\ \sigma'^{11} \end{pmatrix} \quad (2.15)$$

Again, an eigenvector equation for this matrix can be solved and the eigenvalues used for interpretation. A short test in Collischon et al., 2024, also shown in appendix A,

concluded that the mid eigenvalue is suitably sensitive to 4-fold symmetries. Directional information can be taken from the eigenvectors of σ , pulling one index down by multiplying with g as for rank 2 (Collischon, priv. comm.). Note that this method may seem arbitrary, which is a symptom of the higher-rank MT being harder to interpret and other methods may also give sensible results.

Rank 1 MT are zero for closed contours in flat space (Hadwiger and Meier, 1973). Kochappan et al., 2021 used them in the context of CMB analysis by integrating over the field gradient in the search for preferred directions. In the framework presented in Collischon et al., 2024 and here this approach corresponds to $W_1^{0,1}$ calculated for contours given at several brightness thresholds in a window. For small windows, $W_1^{0,1}$ then directly gives the average normal vector direction of the contours contained within. For a closed curve surrounding an area on the sphere the result may also be non-zero due to distortions from parallel transport.

As a sidenote from Euclidean space, the curvature centroids $p_\nu := W_\nu^{1,0}/W_\nu$ can be used to indicate some types of asymmetry (Beisbart et al., 2002, and references therein).

2.2.2 Irreducible MT

As we have seen in the previous section, using the Cartesian MT for data analysis gets non-trivial at higher orders since the morphological information is encoded in their numerous components. Defining the MT in a different representation circumvents this problem for $W_1^{0,b}$.

Let $\rho_K(\varphi)$ be the normal density of K in direction φ , that is, the amount of K 's surface facing direction φ (see Fig. 2.3). Straight sections of the body's contour correspond to high values of $\rho_K(\varphi)$, and heavily bent curves to minima. The sign of φ may vary in different implementations.

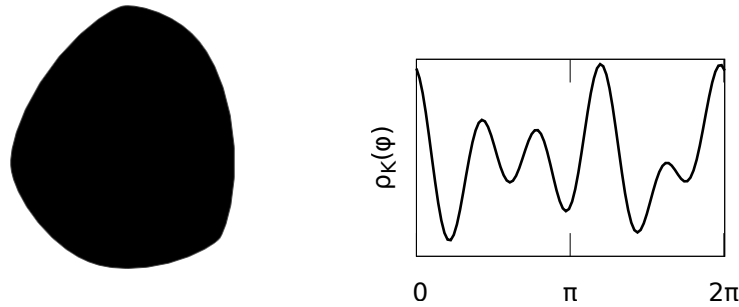


Figure 2.3: “A convex body K and its normal density $\rho_K(\varphi)$ (created with Morphometer 19.04, Schaller, 2019). The minima of the density correspond to the “corner” of the body with high curvature. The maxima correspond to the almost flat regions of the contour.” (Collischon, 2020, Fig. 2.1)

Any shape defined by a pixel image can be considered a polygon. With edges k of length L_k facing direction φ , ρ_K can be written as a discrete sum:

$$\rho_K(\varphi) = \sum_k L_k \delta(\varphi - \varphi_k) . \quad (2.16)$$

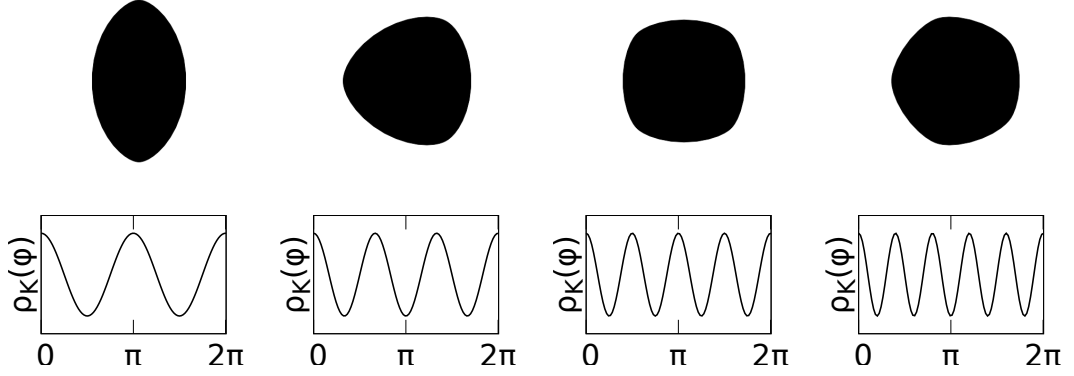


Figure 2.4: “Shapes with $\psi_s \neq 0$ for exactly one s and phase zero (created with Morphometer 19.04, Schaller, 2019). Bottom: Corresponding normal densities (arbitrary units). From left to right: $s = 2, 3, 4, 5 [\dots]$. Since the ψ_s are Fourier coefficients of $\rho_K(\varphi)$, here the ρ_K are given by harmonic waves with increasing frequency.” (Collischon, 2020, Fig. 2.2, four leftmost columns)

ρ_K is 2π -periodic, so it can be discretely Fourier transformed

$$\rho_K(\varphi) = \frac{1}{2\pi} \sum_j \psi_j(K) \exp(-ij\varphi) \quad (2.17)$$

with Fourier coefficients ψ_j , $j \in \mathbb{N}$, given by

$$\psi_j(K) = \int_0^{2\pi} \rho_K(\varphi) \exp(ij\varphi) d\varphi, \quad (2.18)$$

or in the discrete case

$$\psi_j(K) = \sum_k L_k \exp(ij\varphi_k). \quad (2.19)$$

The resulting ψ_j are the so-called irreducible MT. Their amplitude directly gives the amount of j -fold anisotropy found in the shape and their phase contains the corresponding direction (see Fig. 2.4, see also Collischon et al., 2021). They are sometimes normalized with the the total contour length, in which form they are referred to as *Minkowski structure metrics*.

Rotating the body by an angle α results in a phase shift of $j \cdot \alpha$:

$$\psi_j(O(\alpha)K) = \sum_k L_k \exp(ij(\varphi_k + \alpha)) = e^{ij\alpha} \sum_k L_k \exp(ij\varphi_k) = e^{ij\alpha} \psi_j(K). \quad (2.20)$$

Thus, the j preferred directions of a shape for j -fold symmetry are given by

$$\frac{\arg(\psi_j)}{j} + n \cdot \frac{2\pi}{j}, \quad n \in \{0, \dots, j-1\}. \quad (2.21)$$

2.2.2.1 Adaptation to the sphere

The irreducible MT have been brought onto the sphere with the method presented in this subsection for the first time in Collischon et al., 2024.

On the sphere, φ (not to be confused with the coordinate ϕ) is calculated by the direction $\vec{n}$ is facing. Zero corresponds to south (rising θ) and $\pi/2$ to east (rising ϕ), in compliance with the direction of the common unit vectors on the sphere.

Instead of polygons, on the sphere one usually refers to polytopes with geodesical edges. An analogon to (2.19) is used with one caveat: In order for a single angle/normal vector to sensibly describe a segment, the segment needs to be sufficiently short. Otherwise, the angle varies noticeably over the length of the segment. This effect is strongest in the vicinity of the poles.

ρ_K and in turn $\psi_j(K)$ can then be calculated accordingly after parallel transporting all normal vectors to a common point. For L_k the corresponding length of each contour segment is used.

A different, arguably more mathematically careful, approach bringing the irreducible MT to the sphere is presented by Happel et al., 2025. The problems with the above ansatz stated there are, however, vanishing for short contour segments the rest of my work will deal with.

2.3 Minkowski maps

As stated throughout the last section, the MT can be applied to pixel images. To this end, a few concepts are introduced here.

Since the MT can contain local information it is intuitive that they are displayed in the form of maps, where each map pixel contains information about some local neighborhood. Due to these properties, marching square techniques are commonly used (Mantz et al., 2008; Schröder-Turk et al., 2010; Klatt et al., 2012; Collischon et al., 2021).

First a threshold is applied in case the data is not binary but rather given in continuous values, as it is commonly the case for image data. Everything above the threshold is then considered as part of the shape to be analyzed.

Then, a small 2×2 pixel marching square is moved across the image (Fig. 2.5a). There are 16 possibilities of the pixels in a window being either above or below the threshold (see Fig. 2.5b). These 16 cases define how a short contour segment is passing through this region, whose the exact position is interpolated based on the pixel and threshold values. For pixels denoted by A and B with pixel values $p_{A/B}$ and threshold value t , the distance d between the contour segment and A can be calculated according to Mantz et al., 2008

$$d = \alpha \frac{t - p_A}{p_B - p_A}, \quad (2.22)$$

where α is the distance between the pixel centers (see Fig. 2.5c for a visual representation). In the two cases where pixels diagonally across each other are above threshold, the shape is assumed to be connected if the mean value of all four pixels is above the threshold, and disjoint if not (Appleby et al., 2018). Based on this marching square simplification,

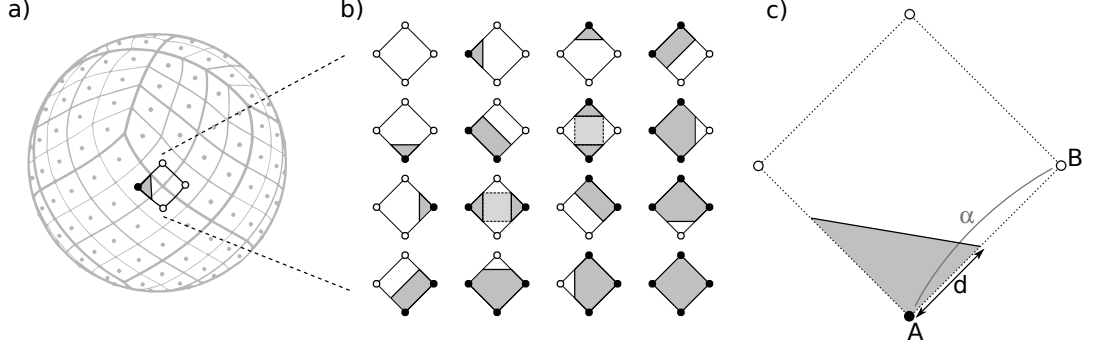


Figure 2.5: “The computation of a marching square on HEALPix. a) Position of a marching square window within the HEALPix grid (HEALPix background image vectorized from Martinez-Castellanos et al. Martinez-Castellanos et al., 2022, Fig. 1c, CC BY 4.0). b) Possible configurations of a 2×2 pixel-window. The small circles represent pixel centers that are either below (open) or above (filled) a certain threshold. The union of gray areas approximates the unpixelated shape within the observation window. In the diagonal cases, the corners are assumed to be disjoint if the total average lies below the threshold, and connected otherwise (Appleby et al., 2018). c) Interpolation of a contour segment (solid line) between pixel centers, where A marks the center of a pixel above the threshold, and the other circles mark pixel centers below the threshold. The point where the interpolated contour lies between two pixel centers is calculated according to eq. 2.22. Here, α is the angular distance from A to B and d the angular distance from A to the interpolated contour in that direction.” (Collischon et al., 2024, Fig. 2)

MT of the segment can be calculated. In the case of masked data, the segments in masked regions may be simply disregarded.

Since such short segments individually contain no meaningful information, a larger window is then used to add up the MT of the underlying short sections, using the additivity property. Information at several threshold values can be combined by calculating the contours for each value and adding them up in the same way. The summation must occur before transforming the (Cartesian) MT into scalars, since the latter operations are often not linear. Additionally, on the sphere the tensors from different places must be parallel transported to a common point to sensibly compare their relative alignment. From these larger windows an output map is then created, where each pixel contains information at the scale of the larger window size.

Software packages to calculate Minkowski maps in the Euclidean plane are, e.g., available in the form of the **papaya2** library for a general toolset (Schaller et al., 2020) and the related **banana** package for analyzing regular images in the fits-format, including automatic detection of bubble-like structures (Collischon et al., 2021).

2.3.1 Minkowski maps on HEALPix

This subsection has been published in similar form in Collischon et al., 2024.

N_{side}	max pixel rad. [rad]	max pixel rad. [deg]
16	0.0660	3.78
32	0.0332	1.90
64	0.0167	0.954
128	0.00834	0.478
256	0.00417	0.239
512	0.00209	0.120
1024	0.00104	0.0598
2048	0.000522	0.0299

Table 2.1: Maximum distance between a pixel center and its corners for the most common values of N_{side} .

All-sky data on the sphere in astronomy and cosmology widely makes use of the Hierarchical Equal Area isoLatitude Pixelation (HEALPix) scheme (Górski et al., 2005)¹. Notably, CMB data by the Planck Collaboration is distributed in this format. Its pixels are located on rings of constant latitude which permits a fast calculation of spherical harmonics. All pixels cover the same surface area, but vary in shape and can be enumerated in either a ring-like or nested scheme. For a visual example of the pixel shapes, see Fig. 2.5a.

The sphere is divided into 12 base squares, four of which are located around the equator and the remaining ones around the polar caps. These are then subdivided into N_{side}^2 smaller pixels, giving a total pixel number of $N_{\text{pixel}} = 12 \cdot N_{\text{side}}^2$. For a sufficient resolution of the smallest structures in the CMB one usually uses $N_{\text{side}} = 2048$.

Pixel shapes vary in a systematic way and so do the average distances between neighboring pixel centers. This is shown in Fig. 2.6, where the average distance from each pixel center to its 7 or 8 neighbors is shown. This variation leads to distortion of contours and care must be taken when using Minkowski maps to compare morphology in different regions. The maximum pixel distances are given in table 2.1.

HEALPix contains 8 spots where only 3 pixels meet in a corner, which needs to be taken into account when implementing Minkowski maps. This creates no fundamental differences; only 8 instead of 16 cases are possible for the contour passing through.

2.3.2 Minkowski analysis with litchi

This subsection has been published in similar form in Collischon et al., 2024.

I implemented the above procedures in a code called `litchi`, available on GitHub at <https://github.com/ccollischon/litchi> and archived at Zenodo (Collischon, 2025). Written in header-only C++ and including Python bindings, it provides fast and versatile access to Minkowski maps. It depends on `libhealpix_cxx`, `pybind11`, and `eigen`; the latter two are included as git submodules and the HEALPix library needs to be installed by the user.

For a low memory usage, it uses an operator template-like structure, where the output map is calculated pixel by pixel directly from the input, so at no point are the raw MT for the whole input kept in memory. This technique may increase computation

¹<https://healpix.sourceforge.io/>

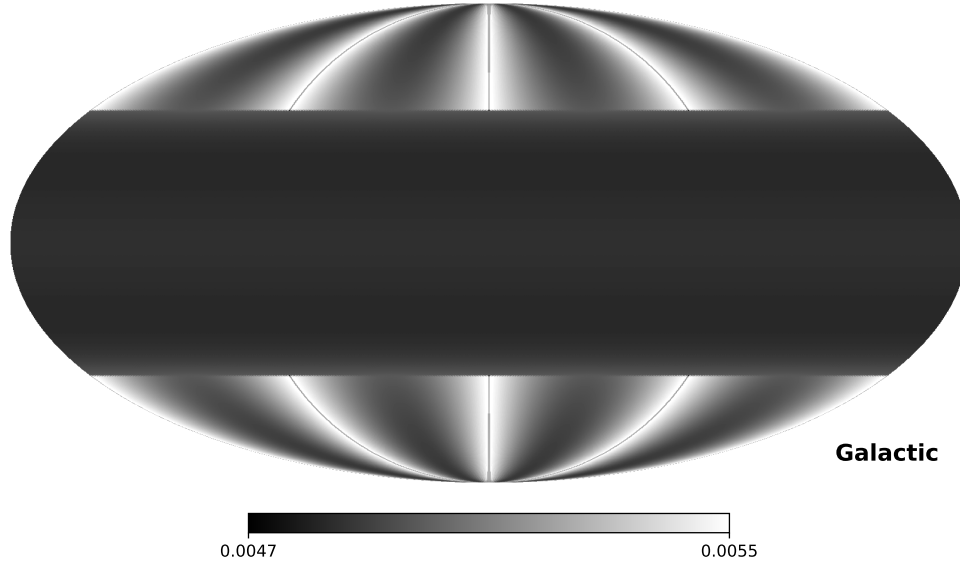


Figure 2.6: “Average distance in radians from each pixel center to its neighbors on the HEALPix grid at $N_{\text{side}} = 256$.” Collischon et al., 2024, Supp. Fig. 4.

time, but avoids hard limitations. Experience has shown that `litchi` profits from modern CPUs with large cache size.

As the 2×2 window is moved across the map to create a Minkowski map, another pixelation scheme is implicitly defined whose pixel centers (corners) are at the corners (centers) of HEALPix pixels, respectively (see the example pixel in Fig. 2.5a). This is not a HEALPix scheme. Internally, `litchi` enumerates such Minkowski map pixels by the HEALPix number of their eastern corner, with special numbers assigned to the polar caps. This takes place purely within the code and the maps are always converted back to HEALPix even when not smoothing any further by assigning each output pixel the average of all 4 bordering Minkowski map pixels.

For all contour segments in the 2×2 windows, the corresponding $\vec{n}$ is calculated at one of the endpoints of the segment. The integration of tensors along a path on a manifold can be expressed as a series (Fabian, 1957). For short segments such as within the marching square, a zeroth-order approximation can be used, where the integrand is merely parallel transported and multiplied with the length of the segment. Since a further analysis is usually done at larger scales, the user chooses a window size of a larger window, to whose center all of these locally calculated tensors are parallel transported and added up.

Parallel transport is implemented using the Schild’s ladder scheme (originally introduced by Alfred Schild in unpublished lectures at Princeton University, presented by Misner et al., 1973, see also Kheyfets et al., 2000). It is an iterative scheme where transport is achieved using steps of a given size. Within `litchi`, only a single step is used to transport the MT calculated for the marching squares to the center of the larger window, where they are added up. Using more steps in a recursive scheme has been tested and made no visible difference while using significantly more resources. Due to the low-order approximations to both tensor integration and parallel transport, it is advisable to use sufficiently large N_{side} .

Masking is achieved by using a mask-file of the same N_{side} and ordering scheme as the input file. A threshold is applied below which all mask pixels are set to be masked (default: 0.9). The input data at these pixels is then set to NAN, and all marching squares touching them produce no contour segment with a corresponding MT. Instead, the would-be tensor is marked as masked. Simply setting the input data to zero (or some other numerical value) would introduce additional contours that affect the outcome. A Minkowski map pixel is set to NAN if more than 1 in 16 of the underlying marching squares are masked. For fewer masked squares, no contours in the masked area are calculated and only the remaining area affects the output.

HEALPix may use either a ring-like or nested numbering scheme, and for `litchi` it is advisable to use the nested scheme as some internal functions execute faster in this setup. Since `litchi` uses operator templates internally, where not the whole map is evaluated at once, but rather output pixel by output pixel, the memory usage remains on the order of the sizes of input and output with little additional overhead. Experience has also shown that it profits tremendously from modern CPUs with larger cache.

`litchi` provides functionality to compute amplitude and direction of anisotropy at any rank for the irreducible MT and at ranks 1, 2, and 4 for the Cartesian MT.

2.3.3 Examples of Minkowski maps

This subsection has been published in similar form in Collischon et al., 2024. Here, however, I expand the discussion of rank 4 and additionally discuss rank 3 and example maps of the phases.

To give the reader a more intuitive sense of the MT displayed as Minkowski maps, a few examples will be presented in this subsection. Fig. 2.7 shows rank 2 Minkowski maps of several example shapes with a given window size for both the Cartesian and the irreducible approach. One can see that each shape causes a characteristic pattern as the window is moved across. Looking at the three rectangles (top left and right and bottom left panels), when the window is centered over the shape the anisotropy is proportional to the aspect ratio (minimal for the square, higher for the other two). Moving to the side, the opposite side of the shape moves out of the window, increasing the anisotropy. Further out, as only a short contour segment is left in the window, the EVQ diverges as one of the eigenvalues goes to zero, but the irreducible anisotropy decreases due to its proportionality to the contour length. For the cross (bottom right panels), one gets a complex pattern of lower anisotropies as different parts move in and out of the window.

Further Minkowski maps for rank 4 are shown in Fig. 2.8, where again both the Cartesian and irreducible forms are shown.

The remainder of this subsection is not presented in Collischon et al., 2024.

Looking closely at the irreducible form, it is identical to the contour length since all contours are perpendicular to each other; no part is canceled out and spherical distortions are negligible at this scale. One can check the validity of the irreducible description by additionally calculating it for other shapes, such as a triangle and a parallelogram. Fig. 2.9 shows Minkowski maps for those two shapes at ranks 4 (top) and 0 (contour length, bottom). Here, rank 4 visibly differs from the length and takes on very low values.

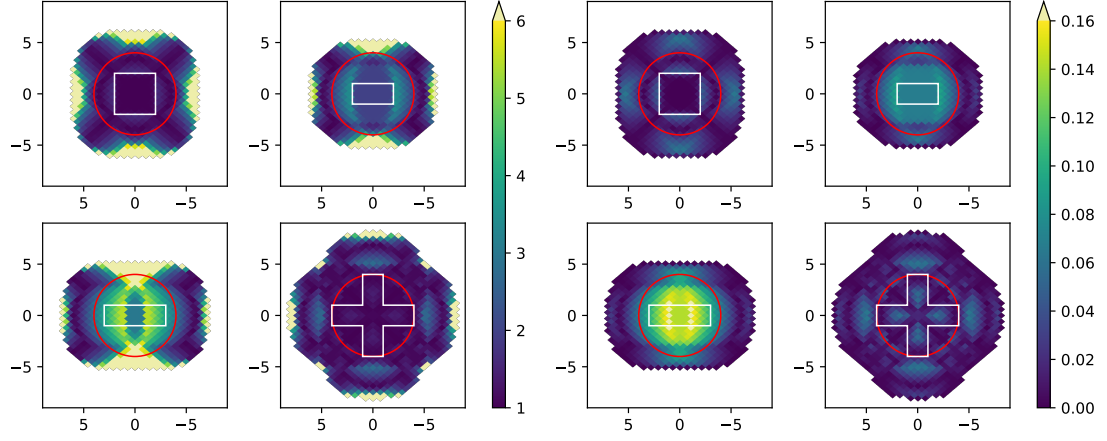


Figure 2.7: Rank 2 anisotropy Minkowski maps of several example shapes (white) for the window size shown in red. Left: EVQ. Right: Irreducible amplitude. Adapted from Collischon et al., 2024, Fig. 3 a & b.

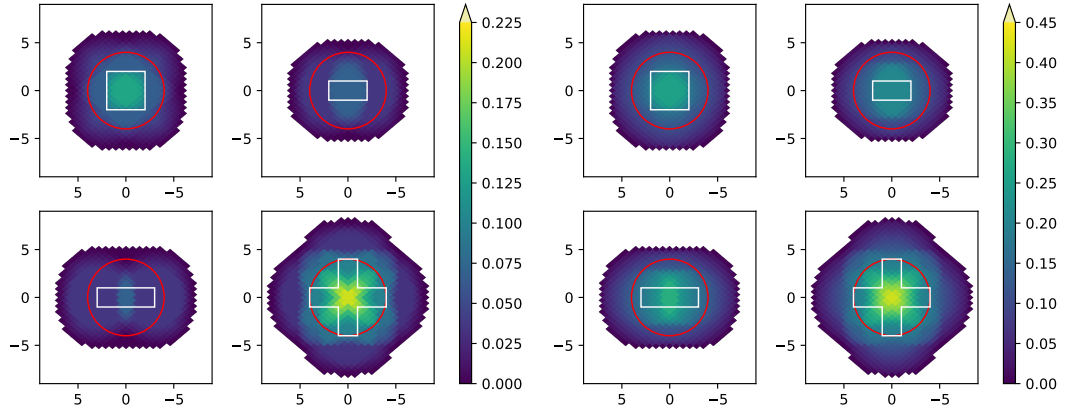


Figure 2.8: Rank 4 anisotropy Minkowski maps of several example shapes (white) for the window size shown in red. Left: Mid eigenvalue according to eq. 2.15. Right: Irreducible amplitude. Adapted from Collischon et al., 2024, Fig. 3 c & d.

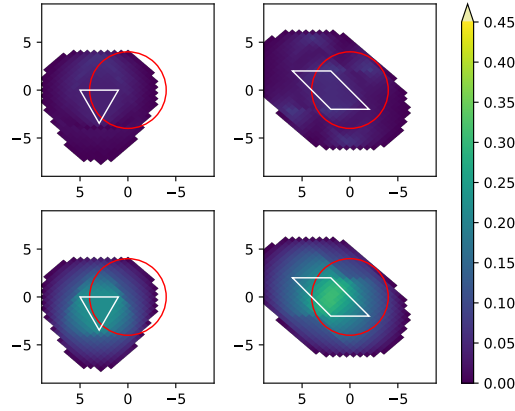


Figure 2.9: Minkowski maps of several example shapes (white) for the window size shown in red. Top: Irreducible rank 4 anisotropy. Bottom: Contour length

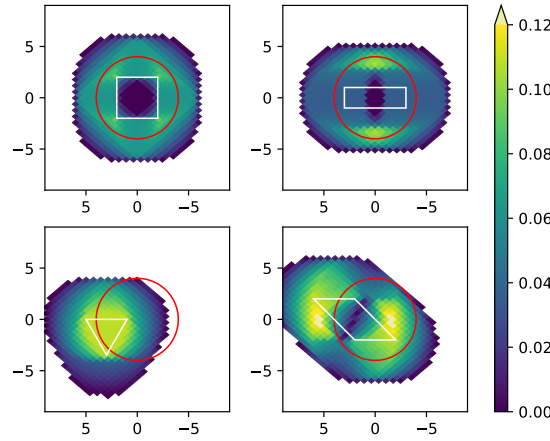


Figure 2.10: Rank 3 irreducible anisotropy Minkowski maps of several example shapes (white) for the window size shown in red.

The same was done for rank 3 (see Fig. 2.10), switching two of the shapes with an isosceles triangle and a parallelogram, since the rank 3 MT should react to triangular shapes. Here, no cartesian anisotropy is available as, to the best of my knowledge, no suitable scalar representation has been found.

As expected, for the two rectangles and the parallelogram $|\psi_3|$ takes low values when the window contains the whole shape and high values for the triangle. When the window covers just a corner of the square or a sharp corner of the parallelogram, the rank 3 anisotropy is high as well.

Surprisingly, the values also rise when moving away from the long side of the long rectangle. This rise can be understood by looking back at (2.19). If only one or a few contour segments at the same angle are contained in the calculation, $|\psi_j|$ is simply the length of the segments for arbitrary j (or unity if one normalizes over the contour length). This is an intrinsic issue of this formalism, since a short straight segment will always add up coherently. The effect is less visible for rank 2 and 4 since it is overshadowed by their intended response to the shape.

Phase and Directions

Additionally, I checked the phases/preferred directions of a few of the example shapes in ranks 2, 3, and 4 (Fig. 2.11). The interpretation of rank 2 is relatively straightforward; it gives the direction in which the shape in the window is elongated. Positive angles refer to counter-clockwise (mathematically positive) rotations of the shape.

For rank 3, one sees that the direction is almost constant near zero for the triangle as its “eigenform”, but rotates around the square and the long rectangle. This rotation in phase cancels out the amplitude at the center of those shapes. Finally, for the parallelogram, both sharp corners resemble triangles when the window only overlaps with one of them. The top left corner has the same orientation as the triangle to its left, leading to a preferred direction close to zero. On the opposite side, the other corner resembles a triangle rotated by 60 deg or $\pi/3$ clockwise. The phase changes continuously around the edges.

In rank 4, the two rectangles appear at constant phase since their contours are horizontal/vertical. Otherwise, the patterns look similar to those found at rank 2, but due to the higher symmetry only preferred directions up to $\pm\pi/4$ can appear. More examples demonstrating the use of the phase at rank 2 to determine orientation are shown in appendix A, Fig. A.5.

2.4 Threshold dependent graphs

If the structure in a given region is to be analyzed in terms of how its morphology depends on brightness, one can create a graph of the chosen structure metric over the threshold. This is an established method of using the MT (see e.g. Planck Collaboration VII, 2020; Joby et al., 2019; Schröder-Turk et al., 2011; Göring et al., 2013; Klatt et al., 2022)

Within `litchi`, the creation of such a graph is usually done by creating Minkowski maps of varying single thresholds of the whole sky and selecting the region of interest afterwards. Several examples are shown in the following chapters (see Figs. 3.5, 3.6, 4.15).

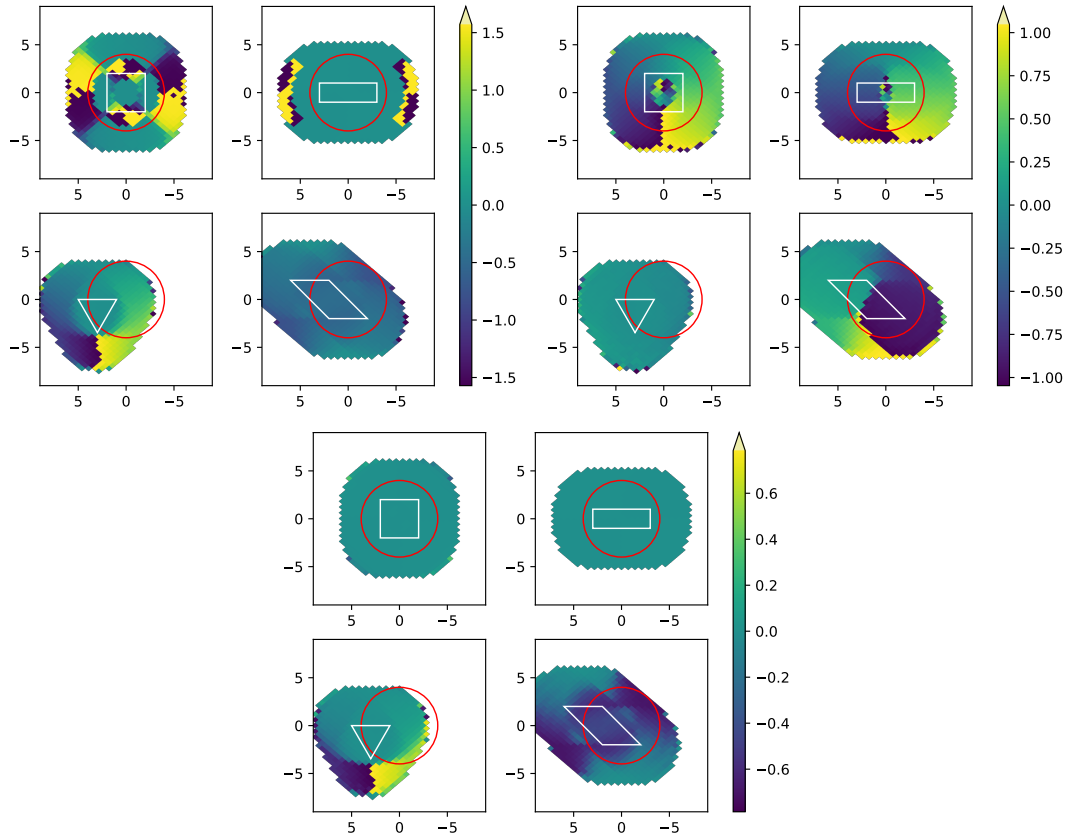


Figure 2.11: Rank 2, 3, and 4 irreducible direction Minkowski maps of several example shapes (white) for the window size shown in red. The colorbar is normalized to $\pm\pi/j$, which represents the real-space preferred directions (modulo $2\pi/j$) of the given rank on the sphere. This corresponds to $\arg(\psi_j)/j$.

Chapter 3

Analysis of Planck data

One important target for a localized analysis with MF are the CMB temperature perturbations (see sec. 1.5), specifically the search for non-Gaussianity. This is often done via comparison with simulations made for a specific cosmology. In this chapter I showcase applications of the MF to the search for local anisotropies in Planck data, presenting various ways to use Minkowski maps.

CMB analysis is an exciting topic, and as such one needs exciting meals along with it. This can be achieved with classic banana pizza, a recipe for which has been previously published in Collischon, 2020.

3.1 Previous works

MF are an established tool for CMB analysis in the search for anisotropies and deviations from Gaussianity as already hinted at in sec. 1.6.1. Notable works are mentioned here.

An early theoretical study was done by Gott et al., 1990 to test the capabilities of the MF in constraining CMB Gaussianity. A first more thorough analysis using all MF was done by Schmalzing and Gorski, 1998. They consider the functionals' calculation in spaces of constant curvature and estimate them for pixelated maps, applying them to COBE data as a proof of concept. Matsubara, 2010 derive the MF analytically for fields with second-order non-Gaussianity. Hikage and Matsubara, 2012 apply this method to WMAP data and find no non-Gaussianity. Buchert et al., 2017 extend their analysis and apply it to Planck 2015 data, finding possible signs of weak non-Gaussianity, which may be affected by residual foregrounds. Modest et al., 2013; Modest et al., 2014 compare WMAP and Planck data to surrogate maps where phases have been shuffled to search for higher order correlations using MF. They find hints for phase correlations at large angular scales on hemispheres that are alleviated when subtracting a specific Bianchi template. A more localized analysis with MF was performed by Novaes et al., 2016, who use them to compare simulated CMB maps to data in 192 regions corresponding to HEALPix pixels at $N_{\text{side}} = 4$. They identify 13 anomalous regions with deviations above 2.2σ , some of which can be explained by hot or cold spots and some by residual foregrounds. More recently, Carones et al., 2023 published the tool **Pynkowski** to calculate expectation values for the MF of Gaussian fields and to analyze CMB polarization data, and Chingangbam and Rahman, 2024 developed analytic formulae for the MF on sums of Gaussian and mildly non-Gaussian fields. Furthermore, the MF

have been used by the Planck Collaboration in the search for non-Gaussianities (Planck Collaboration VII, 2020), detecting no departures from Λ CDM.

Concerning the rank 2 MT, Chingangbam et al., 2017 define $W_2^{1,1}$ on the sphere and give expectation values for Gaussian fields. Their ansatz is used by Joby et al., 2019 to compare $W_1^{0,2}$ statistical anisotropy of both SMICA and single frequency channel temperature data to simulations. For the 30 GHz channel they find deviations they attest to possible beam effects. The work is continued by Goyal et al., 2020, calculating both alignment and anisotropy in rank 2 for both temperature and polarization data. They use this information to study the effect of weak lensing.

One may notice that most of these works calculate the MT for either the whole sky or hemispheres, the notable exception being Novaes et al., 2016, who only used the MF. This approach may be useful in the search of larger scale non-Gaussianity, but local effects are hard to find - Novaes et al., 2016 report that interesting structures are sometimes divided by their window border. A true search for local features thus requires moving windows evaluated with some overlap, as can be done with Minkowski maps. Additionally, for many of these works the analysis code is not publicly available, unlike `litchi`.

In the remainder of this chapter, I will present ways to analyze Planck CMB temperature data using Minkowski maps.

3.2 Planck data products and HEALPix

A vast amount of Planck data products is publicly available in the Planck Legacy Archive¹. Most importantly for this work, temperature deviation maps using the various foreground cleaning methods and accompanying Full Focal Plane (FFP) simulations are provided. Additionally, a Common Mask used for excluding the most contaminated areas is available.

Data products are usually given at a resolution of $N_{\text{side}} = 2048$ to encompass relevant higher orders. To reduce computation time or to exclude those high orders, the maps are often degraded to a lower resolution using the tools provided in the libraries accompanying HEALPix. To this end, the maps are first transformed into harmonic space, taking pixel window functions into account since pixels cover a finite area of the field (these are included in the `pixwin` function of the `healpy` package). Then the coefficients are deconvolved with the previous beam size and convolved with a larger window size (see Table 1 of Planck Collaboration VII, 2020, for commonly used values, in particular $5'$ for $N_{\text{side}} = 2048$ and $20'$ for $N_{\text{side}} = 512$). The then lower resolution is captured in a map of lower N_{side} . A general didactic overview on CMB map analysis techniques going into more detail on these topics is given by Sullivan et al., 2025.

Since `litchi` operates faster on maps using the nested numbering scheme, it is advisable to apply this scheme when transforming back to the output map. I used $l_{\text{max}} = 3N_{\text{side,out}} - 1$ since spherical harmonics on HEALPix are not independent above this value.

¹`pla.esac.esa.int`

3.3 Minkowski map analysis

The following section follows Collischon et al., 2024 with a few additional figures.

I created Minkowski maps of the PR3 SMICA CMB temperature map downgraded to $N_{\text{side}} = 512$ and smoothed with a FWHM of $20'$. I used a Minkowski window radius of 6 deg, using 13 thresholds between 3 standard deviations below and above the mean pixel value of the map. The same was done for 999 FFP10 simulation maps. All input data was masked with the Common Mask which was degraded and smoothed similarly, setting all pixels below 0.95 to masked. The output $N_{\text{side}} = 16$ to create sufficient overlap between the windows while lowering computation time.

Looking at the EVQ at rank 2, I created maps of the local deviation between data and simulation Minkowski maps. Fig. 3.1 (top) shows this deviation in units of pointwise standard deviation. Standard deviations are intuitive and uncomplicated to visualize, but not a good measure for non-Gaussian distributed data. For a more accurate portrayal of the deviation, I created maps of the local and global percentile of each data Minkowski map pixel with respect to the simulations; that is, the fraction of simulation Minkowski map pixels at the same location or globally with a lower value than the data (see Fig. 3.1 center and bottom). The local percentile has the advantage of taking pixel distortions into account, while the global percentile is of higher statistical relevance overall. Using just the local method outstanding regions in the data may be compared to average regions in the simulations, distorting the result. In order to also detect regions larger than a Minkowski map pixel that might be outstanding, the same maps are also calculated after smoothing the Minkowski maps with a FWHM of 6 deg (Fig. 3.1, right column).

Two spots stand out in all maps, one near $(l_1, b_1) = (225, -69)$ and one near $(l_2, b_2) = (80, 27)$ with more than 4 standard deviations between data and simulation mean or percentile values above 99%. The spots are marked on the CMB map in Fig. 3.2 (l_1, b_1) is close to the Cold Spot around $(l, b) = (210, -57)$ (see sec. 1.6.1). The distance between both is however larger than the window size, so a direct correlation is not assumed. The pixel at $(l_3, b_3) = (337, 62)$ reaches comparable deviation values as (l_1, b_1) in the unsmoothed maps, but here only a single pixel stands out ((l_1, b_1) and (l_2, b_2) are surrounded by several higher pixels). Since it does not stand out in the smoothed maps I will not discuss it further.

Care needs to be taken regarding the look-elsewhere-effect, where notable points in the data are compared to average points in the simulations, which may lead to false conclusions (see also sec. 1.6.1). For a quick estimate of this effect, I created maps counting the number of simulation Minkowski maps where at least one pixel surpasses a given data pixel. (l_2, b_2) is surpassed in only 6 maps, whereas 943 out of 999 simulation maps contain a pixel with a higher value than (l_1, b_1) . At first glance, this seems not very notable. However, since (l_1, b_1) is in the second-highest region in the data Minkowski map, it should ideally be compared to the regions with the second-highest values in the simulations, taking correlations into account. Since the Minkowski map windows overlap at the output resolution used here, neighboring pixels may share similar high values describing the same region. Simply taking the second-highest pixel values in each map will then not be sufficient and a more elaborate method is necessary. Additionally, pixel distortion effects are not taken into account here. This result should thus be taken with a grain of salt.

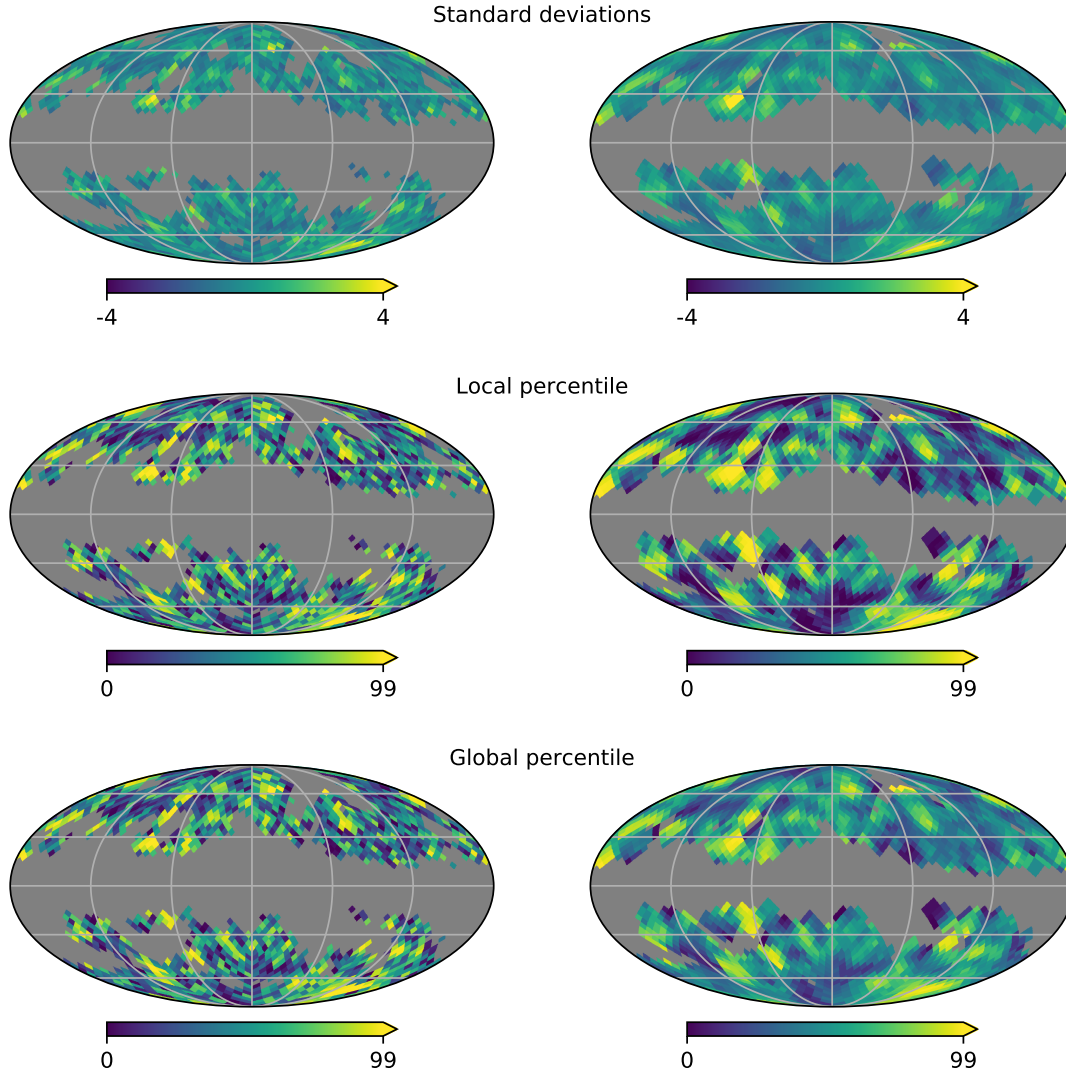


Figure 3.1: Deviation between Planck PR3 SMICA CMB temperature data and FFP10 simulations as measured by rank 2 anisotropy in units of standard deviations (top), local percentiles (center), and global percentiles (bottom). Left: Using unsmoothed Minkowski maps. Right: Using Minkowski maps smoothed with FWHM 6 deg. Input $N_{\text{side}} = 512$, output $N_{\text{side}} = 16$, Minkowski map window radius = 6 deg. Adapted from Collischon et al., 2024, Fig. 4.

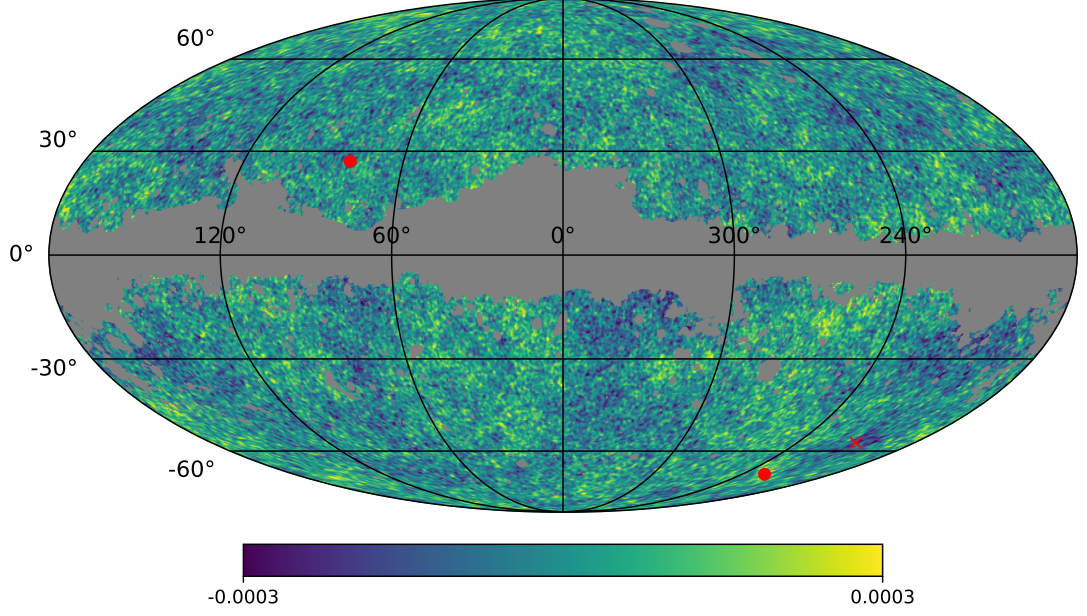


Figure 3.2: SMICA PR3 CMB temperature map at $N_{\text{side}} = 512$ masked with the Common Mask (gray) with the notable spots $(l_1, b_1) = (225, -69)$ and $(l_2, b_2) = (80, 27)$ (red points) and the Cold Spot (red cross). From Collischon et al., 2024, Fig. 5.

Repeating the analysis with data and 999 simulations from the Commander pipeline, the two regions stand out again. Here, the standard deviation of the simulation pixels is larger, lowering the apparent deviation in units of standard deviation. Both regions however still reach high percentiles of $> 99\%$ (local, global unsmoothed maps) and $> 97\%$ (global smoothed maps). The same holds when using the irreducible rank 2 anisotropy instead of the EVQ. To evaluate the effect of pixel distortions, I rotated both regions to the origin and repeated the calculations. Both regions still stand out at $> 3.5\sigma / > 99.4\%$ for (l_1, b_1) and $> 4.4\sigma / > 99.9\%$ for (l_2, b_2) , implying that the observed effect is not strongly affected by the distortions.

Similarly, varying the window radius by around 10% did not change the results, meaning the analysis is not sensitive to this specific setting.

The local surroundings of the spots in the data, data Minkowski maps, and the local percentile maps are shown in Figs. 3.3 and 3.4. One can see that while (l_2, b_2) is near a masked region in the Minkowski maps, only a handful of smaller masked regions in the temperature data are nearby. These spots should not affect the analysis; the Minkowski map pixels appear masked only due to the restrictive masking routine applied by `litchi`.

To further analyze the two spots, I created threshold dependent graphs for the same data and 600 simulations (due to the higher computational cost) for a 6 deg radius around the spots (see Fig. 3.5). The data exceeds 99% of the simulations for (l_1, b_1) at thresholds around 0 to 0.5 standard deviations above mean, but is within the first and 99th percentile for (l_2, b_2) . This lack of discrepancy means that the overall deviation at (l_2, b_2) seen in Fig. 3.1 is highly affected by correlations between contours

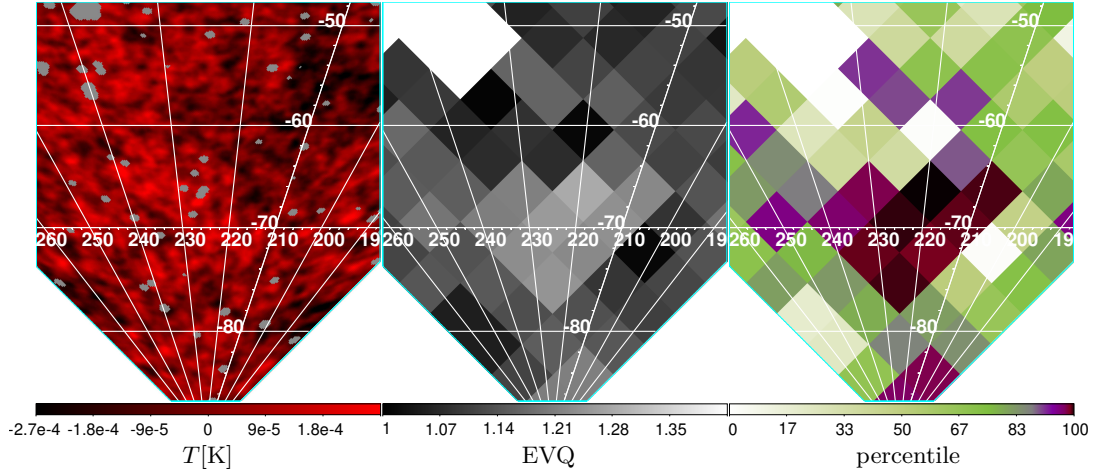


Figure 3.3: The surroundings of $(l_1, b_1) = (225, -69)$ in the temperature data (left, gray areas are masked), the rank 2 Minkowski map with the parameters stated in the text (center), and the local percentile of the unsmoothed Minkowski map (right). From Collischon et al., 2024, Supp. Fig. 2

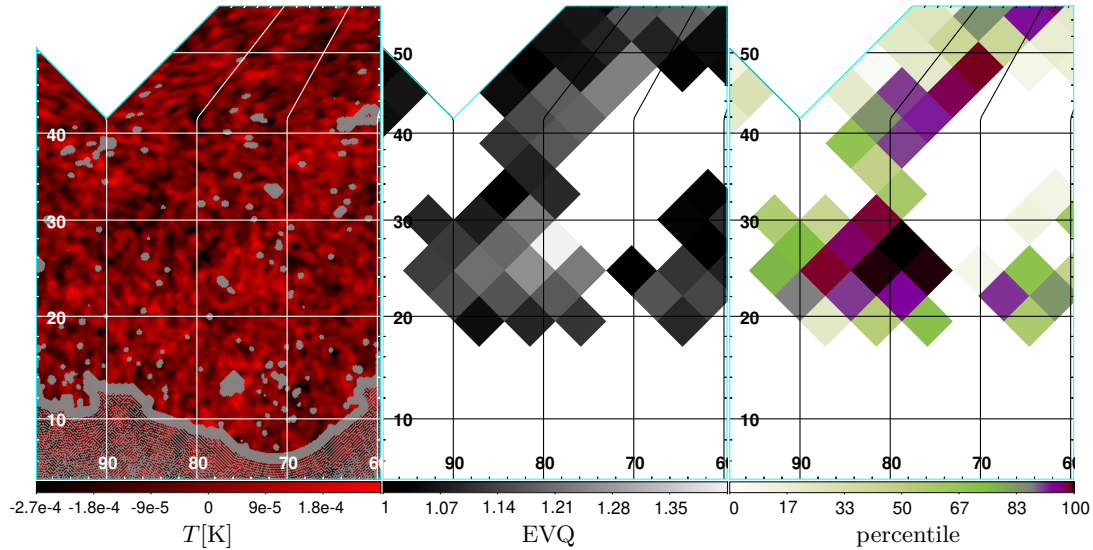


Figure 3.4: The surroundings of $(l_2, b_2) = (80, 27)$ in the temperature data (left, gray areas are masked), the rank 2 Minkowski map with the parameters stated in the text (center), and the local percentile of the unsmoothed Minkowski map (right). From Collischon et al., 2024, Supp. Fig. 1

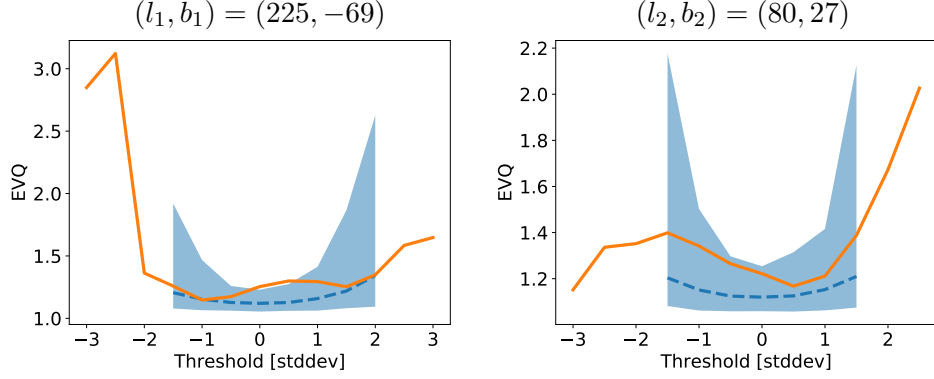


Figure 3.5: Threshold dependent graphs of the EVQ of the regions around (l_1, b_1) (left) and (l_2, b_2) (right) showing data (orange solid line) and simulation (blue) anisotropy at various thresholds in units of temperature standard deviation. For the simulations mean (dashed line) and percentiles between 1 & 99% (light blue area) are shown.

at different thresholds. Minkowski maps combining multiple different thresholds thus contain additional shape information².

A similar analysis has been done by Joby et al., 2019, who created threshold-dependent graphs of rank 2 anisotropy of the whole sky for both SMICA temperature data and single frequency channels. They follow a different mathematical approach to this calculation specific to rank 2 tensors and compare the data to 100 FFP8 simulations. The temperature data and simulations are in good agreement according to their method, but for the 30 GHz channel alone they find that the data is more anisotropic than the simulations by about 2σ .

For a rough comparison to their results, I created threshold dependent graphs of the whole sky for a Minkowski window size of 6 and 10 deg (see Fig. 3.6). The data is around the lowest percentile of the simulations in the 10 deg maps for thresholds close to the average temperature. This may point to the same alignment of structures Joby et al., 2019 found in the 30 GHz channel. They attest this behavior to beam effects.

Additionally, I created threshold dependent graphs for the northern and southern Galactic hemispheres for EVQ Minkowski maps with a window size of 6 deg, where again the data is around the lowest simulation percentile for thresholds close to the average temperature (Fig. 3.7). The data is within the simulations for both hemispheres, but whereas in the southern part the data follows the simulation mean, in the northern part there is more deviation (see also sec. 1.6.1). To check whether this deviation is within the limits, one would need to look at the north-south difference in all simulations.

The above discussions were all restricted to relatively local structures at a size of 6 to 10 deg. Minkowski maps can however be created for larger window sizes. I calculated the above deviation maps also for a radius of 30 deg, where due to restrictive masking around the Galactic Plane only information around the polar caps is available. At these

² *Additional information not discussed in Collischon et al., 2024:* One can see in Fig. 3.5 that the simulation values are not depicted for large and small thresholds. The mean and percentile values are NAN here since in at least one simulation map there were not sufficient contours for a Minkowski analysis.

This is expected for a small region analyzed for single thresholds. The rising variance in the simulations towards large absolute threshold values is another testament to a falling contour length.

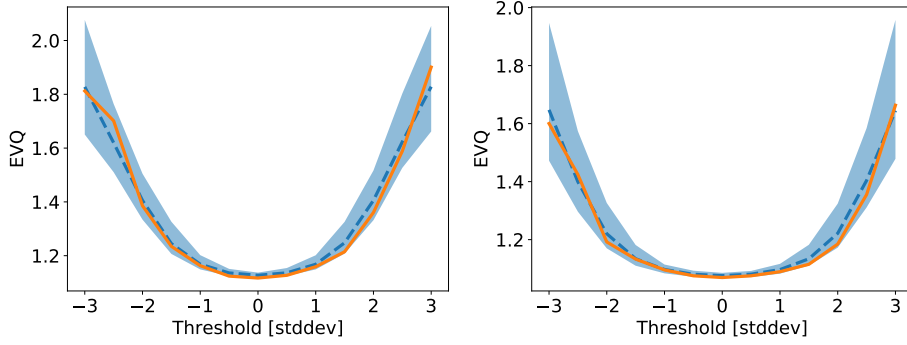


Figure 3.6: Threshold dependent graphs of the EVQ of the whole sky for Minkowski window size of 6 deg (left) and 10 deg (right) showing data and simulation anisotropy at various thresholds in units of standard deviation. For the simulations mean (dashed line) and percentiles between 1 & 99% (light blue area) are shown.

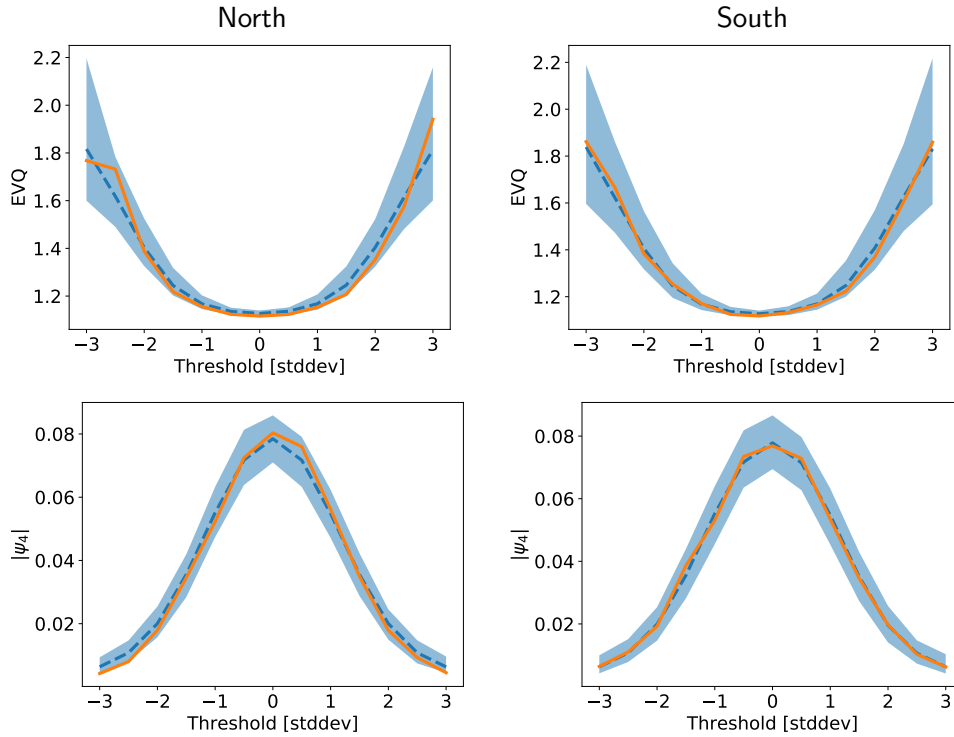


Figure 3.7: Threshold dependent graphs of the EVQ (top) and $|\psi_4|$ (bottom) of the northern (left) and southern (right) Galactic hemispheres for Minkowski window size of 6 deg showing data (orange solid line) and simulation (blue) anisotropy at various thresholds in units of temperature standard deviation. For the simulations mean (dashed line) and percentiles between 1 & 99% (light blue area) are shown.

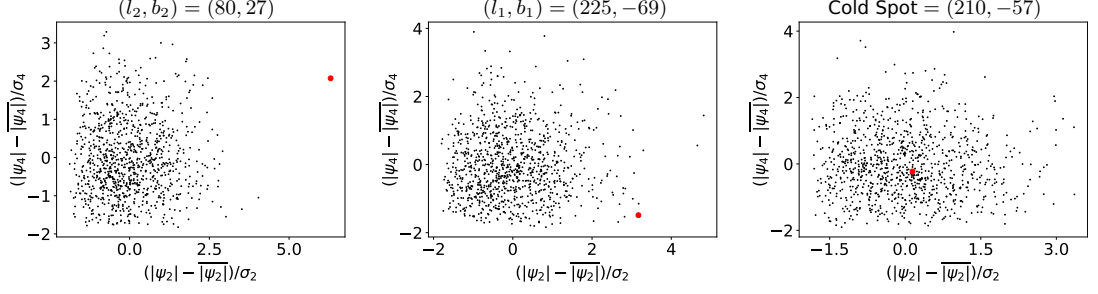


Figure 3.8: Scatter plots of rank 4 and rank 2 irreducible anisotropy for pixels at the noteworthy spots (l_2, b_2) (left), (l_1, b_1) (center), and the Cold Spot (right). Black small dots: Simulation Minkowski maps, Red large dots: Data. Adapted from Collischon et al., 2024, Fig. 6.

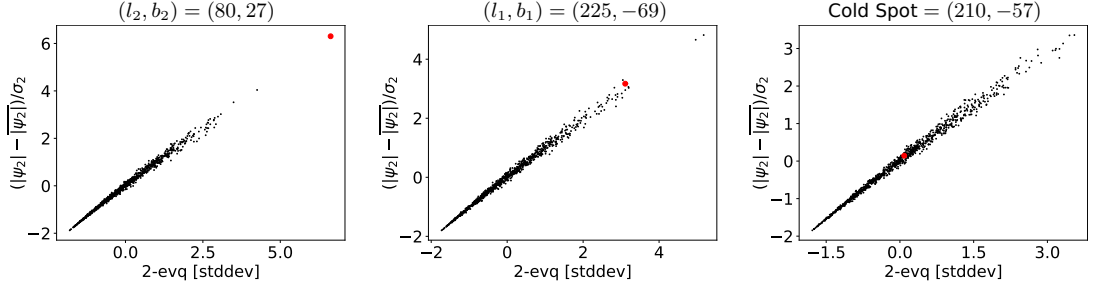


Figure 3.9: Scatter plots of EVQ and rank 2 irreducible anisotropy for pixels at the noteworthy spots (l_2, b_2) (left), (l_1, b_1) (center), and the Cold Spot (right). Black small dots: Simulation Minkowski maps, Red large dots: Data.

scales the data does not deviate far from the simulations ($< 98.7\%$ of simulations both globally and locally).

In order to scout the possibilities of higher ranks, I created scatter plots of pixels at both spots showing irreducible rank 2 and 4 anisotropies (see Fig. 3.8). For a reasonable comparison, the values are normalized using the local mean and standard deviation. The plots show that these different ranks are not correlated in any way visually detectable and thus contain independent information. One sees that in rank 2 at (l_2, b_2) , the data is far from the simulation pixels, at (l_1, b_1) it surpasses most simulation pixels being located at the edge of the distribution, and at the Cold Spot the data is close to the average of the simulations. In rank 4 only (l_2, b_2) is somewhat noteworthy.

The following extended analysis is not part of Collischon et al., 2024.

Additionally to the above, I added Gaussian noise with a standard deviation of 0.1 of the temperature map's deviation to both data and simulations before degrading to $N_{\text{side}} = 512$. Both (l_1, b_1) and (l_2, b_2) still stand out at similar values as before.

To confirm the equivalency of the irreducible anisotropy at rank 2 and the EVQ, I created scatter plots like in Fig. 3.8 for $|\psi_2|$ and the EVQ (Fig. 3.9). One can easily see that the two anisotropy measures are linearly correlated. Such a behavior is expected since they are both constructed to contain information about elongation.

3.4 Discussion

If the spots around (l_1, b_1) and (l_2, b_2) can be confirmed in other data and simulations, they would add to the pile of possible anisotropies in the CMB temperature field (sec. 1.6.1). As opposed to more unintuitive issues such as the lack of large angle correlations, these anomalies have the direct implication of the universe producing more elongated structures localized around those directions. Outside of these spots however, I can confirm the same level of Gaussianity as in the simulations as measured by rank 2 anisotropy.

Another avenue to look into are the regions of low anisotropy, where the data is more isotropic than the simulations. The analysis could be follow the same principles as above. The same holds for MT of higher rank. Here one needs to be careful not to overtest; for many independent measures the probability of finding an outlier by pure chance rises. I therefore leave the analysis at the level presented here, having shown the capabilities of the MT in CMB analysis.

Chapter 4

Large-scale structure

MT can also be used to look at LSS data as another cosmological probe. The evolution of structure morphology over time can give insight into cosmic assumptions and parameters. Similarly to the CMB, one may also search for statistical outliers unexplained by models. Being usually made up of point sources, the data sets need to be made accessible to the MT first. Over the years, several possible ways to tackle this issue have been found.

If the reader, at this point of reading this thesis, is in need of caffeine, I recommend Rösttrommel’s “Edgar Bastianello” as an excellent and deeply flavorful choice when prepared as an espresso or cappuccino, going well with the subsequent chapter.

4.1 Previous analyses with Minkowski functionals and tensors

An early application of MF to 3D point patterns has been presented by Mecke et al., 1994. A ball is placed at the center of each point, leading to different topological patterns depending on the radius. They then compare a Poisson-process, a ‘double-Poisson’ process and the ACO galaxy cluster catalog (Abell et al., 1989). Their results show that different processes can be robustly distinguished using the MF. Further applications of this style are found in (e.g.) Kerscher et al., 2001a; Kerscher et al., 2001b.

Other works (e.g. Grewal et al., 2022; Jiang et al., 2024, and references therein) calculate the MF directly on density fields using field derivatives. They smooth the fields with a Gaussian kernel to mitigate shot noise. Grewal et al., 2022 find that while the information contained in the MF is limited, using them in combination with the power spectrum improves constraints on small-scale effects. However, this approach to the calculation of MF is affected by masking, which should not be a notable issue when using `litchi` instead. In terms of higher ranks, other works applied the MT to redshift-space distorted fields (Appleby et al., 2023; Appleby et al., 2025).

4.2 This work

For an analysis with `litchi`, point source distributions must be projected onto the spherical surface in HEALPix. 3D Simulations cannot be used directly, but rather lightcones that take redshift evolution and observational effects into account need to

be calculated. Then, one can simply take the projected coordinates of each object and create a density map in HEALPix at the desired N_{side} .

A map where only single pixels have nonzero values cannot sensibly be used directly for the creation of a Minkowski map. Therefore, in the spirit of previous analyses, I smooth these maps with a Gaussian filter of a chosen FWHM. Larger FWHM merge nearby sources into a single blob, whereas smaller ones may leave sources almost completely disjoint. Then, choosing a single threshold in combination with the FWHM corresponds roughly to choosing an extent for each point source. Neighboring point sources are added up, creating an overall larger extent. Still, each point source pattern corresponds to a characteristic map that can be analyzed with the methods at hand. Creating a threshold dependent graph this way is a close analogy to previous works. Alternatively, one can combine contours at several thresholds and analyze those Minkowski maps like in Chapter 3.

I apply this smoothing technique to various data and simulation sets. The aim of the following analyses is not primarily to extract scientific output from the data sets, but rather to explore methods, showcase proofs-of-concept, and to analyze the behavior of `litchi` under different conditions. With this in mind, there are many possible ways to analyze each data set and one would have to choose the best method for a task at hand. The subsequent sections can be seen as examples of different approaches for future, more thorough analyses.

4.2.1 Uchuu-SDSS/GLAM

One source of point source data and simulations is provided by Dong-Páez et al., 2024, who published simulations and a corresponding selection of data from the seventh data release of the Sloan Digital Sky Survey (SDSS) (York et al., 2000; Abazajian et al., 2009).

Their first set of simulations, Uchuu, comprises 32 high-resolution, high-fidelity LSS simulations using Planck cosmology. The Uchuu-SDSS galaxy catalogs are built by using SDSS galaxy data to populate the simulated matter distribution. Galaxy lightcones are available to enable comparison to observational data. Additionally, they provide 5100 lower resolution GaLaxy Mocks (GLAM) simulation galaxy lightcones (Klypin and Prada, 2018) and compare all the simulations to a selection of galaxies in the NYU Value Added Galaxy Catalog, which is a subset of the SDSS (Blanton et al., 2005). This selection is restricted to regions with $> 90\%$ completeness in the contiguous northern footprint, containing $\sim 497\,000$ galaxies over an effective area of $\sim 6511\text{ deg}^2$.

I created point source density maps of the simulations and the data they used, binning them on the HEALPix grid and smoothing them with a Gaussian kernel of varying FWHM. One example of density maps for both types of simulation is shown in Fig. 4.1. Note the different scales for Uchuu and GLAM due to the vastly larger amount of objects in Uchuu.

Since the data are located in a region with systematic effects in sky coverage, masking is necessary. I found no spatial mask provided by the authors and instead created a mask by visually identifying regions with obvious systematics, and including only regions with a smoothed point source density above 10^{-8} at a FWHM of 0.2 deg (see Fig. 4.2). Simply leaving certain point sources out of the density map would not have achieved the same effect, since the output Minkowski map should be set to NAN in

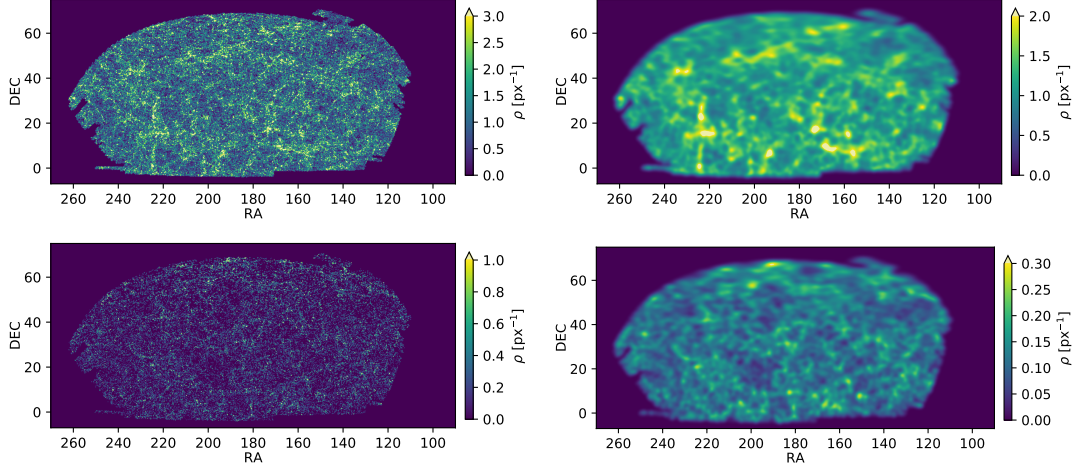


Figure 4.1: Density maps of LSS simulations at $N_{\text{side}} = 512$. Top: Uchuu, bottom: GLAM. Left: unsmoothed point source count per pixel, right: smoothed with FWHM 2 deg.

regions with high uncertainty. The mask is smoothed with the same FWHM as the point source maps to be analyzed. Still, such a “pragmatic” approach to masking is not sufficiently robust for a thorough analysis - diffuse foregrounds may have affected the data in more subtle ways. Thus, the following results are necessarily restricted to proofs-of-concept.

Now, it is possible to create a Minkowski map of the smoothed density field with several thresholds, containing information on the overall structure, akin to previous analyses. The variation in point source radius is achieved here by variation of the smoothing radius. Alternatively, one could create threshold-dependent graphs at a single smoothing radius.

With the former technique, I created rank 2 Minkowski maps of the irreducible rank 2 anisotropy at FWHMs of 2, 1, 0.6, 0.2, and 0.1 degrees with input $N_{\text{side}} = 1024$ and output $N_{\text{side}} = 32$, using a Minkowski window size of 6 deg. I used 13 evenly spaced thresholds, the lowest one 3 standard deviations below the input map’s mean pixel brightness and the highest one 3 standard deviations above. More precisely, I chose a circular window with a radius of 15 deg around $(l, b) = (200, 25.5)$ that encompasses the center of the data to calculate the mean and standard deviations. This window omits most masked regions while still containing a sufficient number of pixels. Jiang et al., 2024 discuss the effect of threshold choice in terms of either the density contrast directly or the normalized density (that is, the density contrast divided by the root mean square of the field as chosen here). Since in this case I am dealing with simulations and data with vastly different number densities, only normalized density thresholds have a chance to give comparable structure information. If you are reading this and are not an examiner, send me a screenshot and I will owe you a drink. From these maps I created deviation maps of the global percentile of the data with respect to both simulation types (smoothed at 2 deg, see Figs. 4.3 and 4.4).

For the varying radii, different spots arise as deviations between data and simulations, demonstrating the MTs’ versatility of finding structure at various scales. Overall,

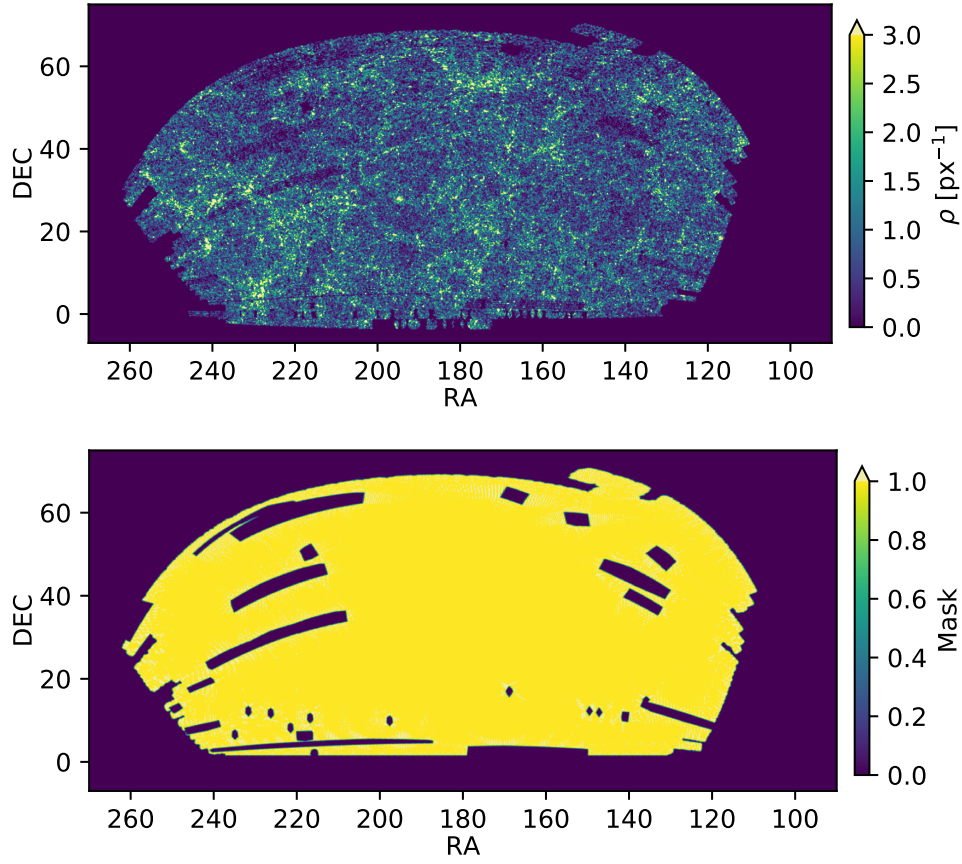


Figure 4.2: SDSS/eBOSS source density map (top) and mask (bottom) at $N_{\text{side}} = 512$.

however, this is a simple analysis with only 400 simulations whose statistical significance should not be overestimated.

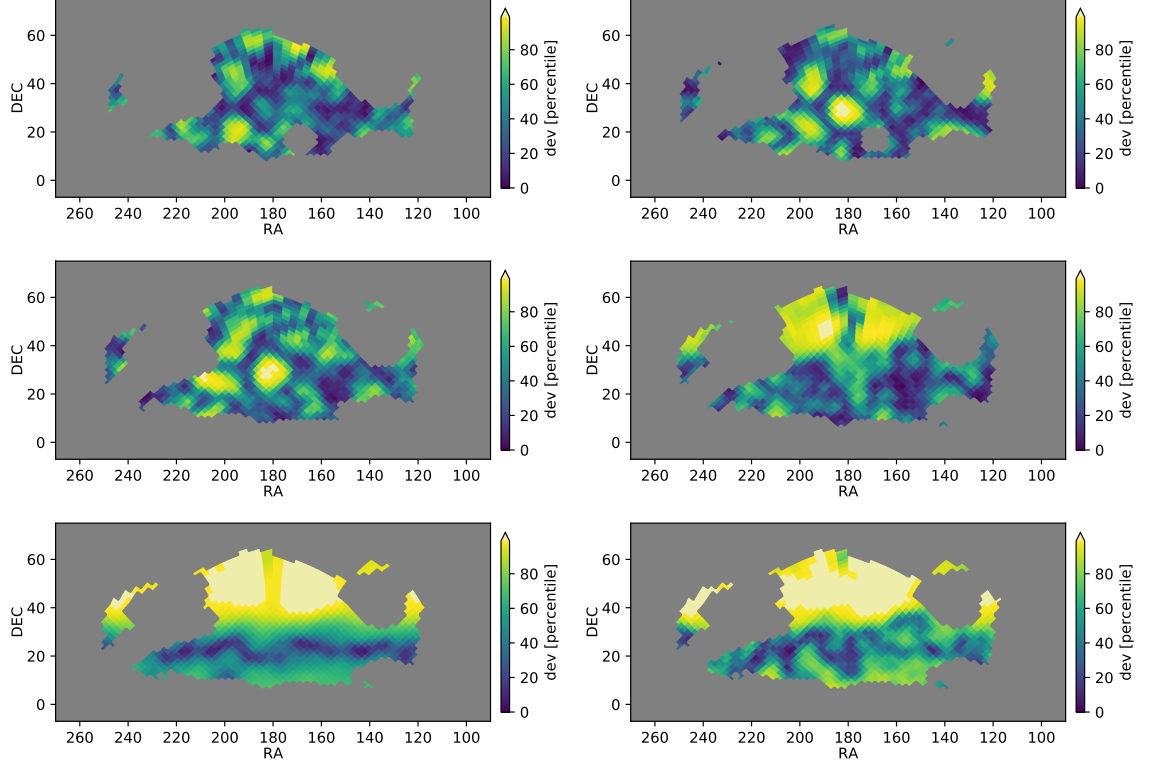


Figure 4.3: Deviation between SDSS data and GLAM simulations as measured by global percentiles of rank 2 anisotropy for Minkowski maps with window size 6 deg. Input point source density maps used $N_{\text{side}} = 1024$ and were smoothed with a FWHM of 2, 1 (top), 0.6, 0.2 (center), and 0.1 deg (bottom left). Bottom right: Input $N_{\text{side}} = 2048$ and FWHM = 0.1 deg.

For the smallest FWHM there seem to be large deviations between data and GLAM simulations in large areas (Fig. 4.3, center right and bottom left) and the same characteristic pattern appears for the Uchuu simulations, albeit with significantly smaller deviation (Fig. 4.4, center right and bottom left).

One effect at play for the GLAM simulations is their significantly lower number of sources and thus at low smoothing scales lower contour length than the data. At smoothing scales large enough to merge most sources this is not an issue since thresholds are placed relative to the overall brightness of each map. Since ψ_2 is proportional to the contour length the source count will otherwise affect the analysis. The effect should however be equally distributed over the data and not show the systematic pattern at hand.

To see if normalizing the GLAM maps with the contour length eliminates the difference I recalculated the deviation maps for the normalized anisotropy $|\psi_2|/\psi_0$ (Fig. 4.5). In this version, the GLAM deviation with respect to the data is reduced, but still significant, and the systematic pattern persists.

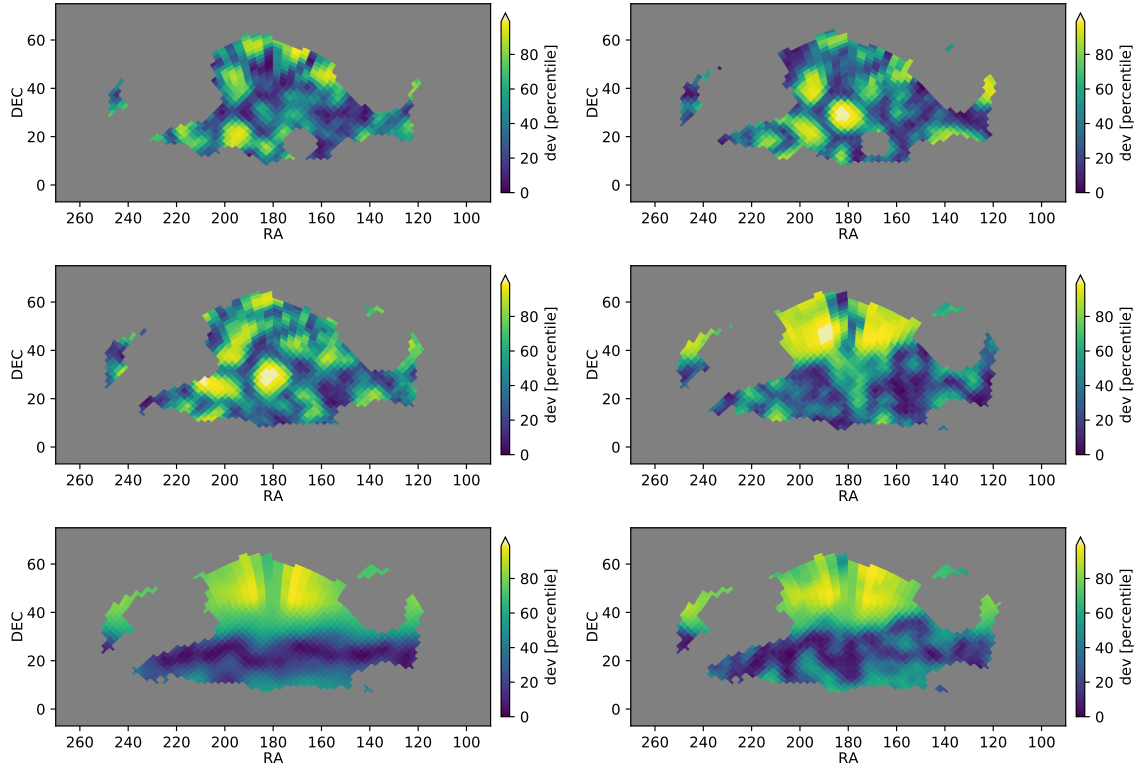


Figure 4.4: Deviation between SDSS data and Uchuu simulations as measured by global percentiles of rank 2 anisotropy for Minkowski maps with window size 6 deg. Input point source density maps used $N_{\text{side}} = 1024$ and were smoothed with a FWHM of 2, 1 (top), 0.6, 0.2 (center), and 0.1 deg (bottom left). For the bottom right map, Input $N_{\text{side}} = 2048$ and FWHM = 0.1 deg.

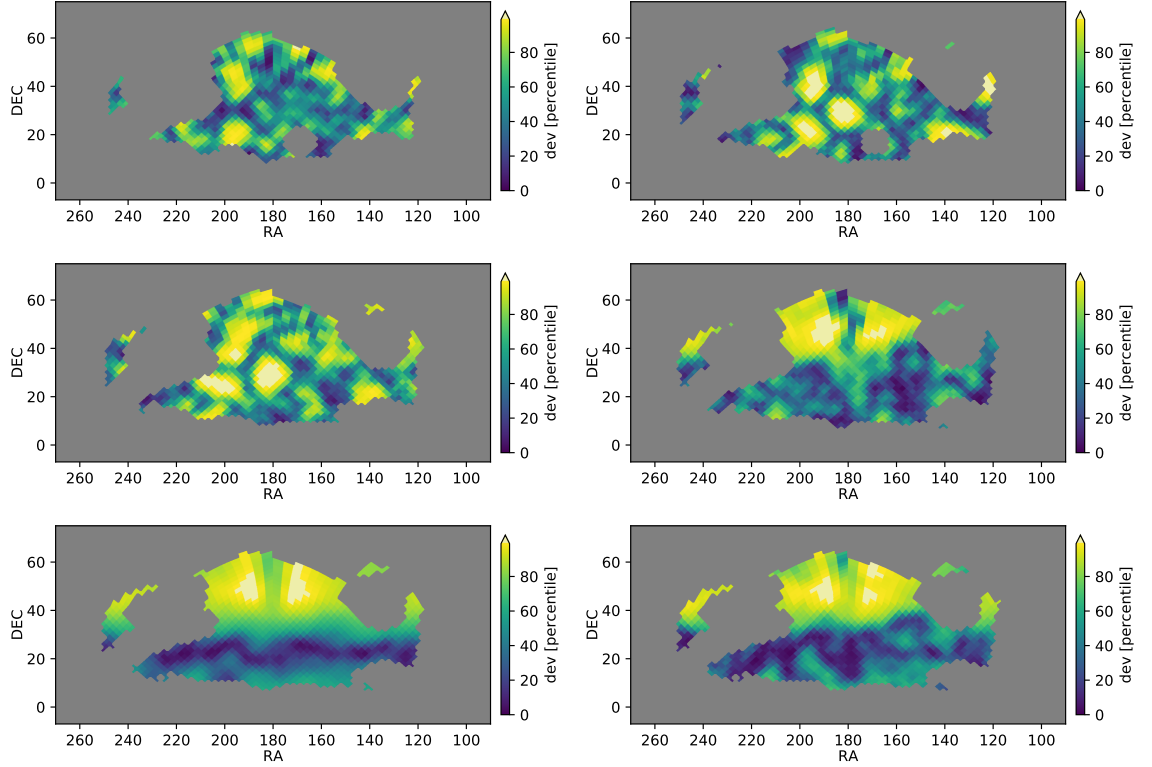


Figure 4.5: Deviation between SDSS data and GLAM simulations as measured by global percentiles of normalized rank 2 anisotropy for Minkowski maps with window size 6 deg. Input point source density maps used $N_{\text{side}} = 1024$ and were smoothed with a FWHM of 2, 1 (top), 0.6, 0.2 (center), and 0.1 deg (bottom left). Bottom right: Input $N_{\text{side}} = 2048$ and FWHM = 0.1 deg.

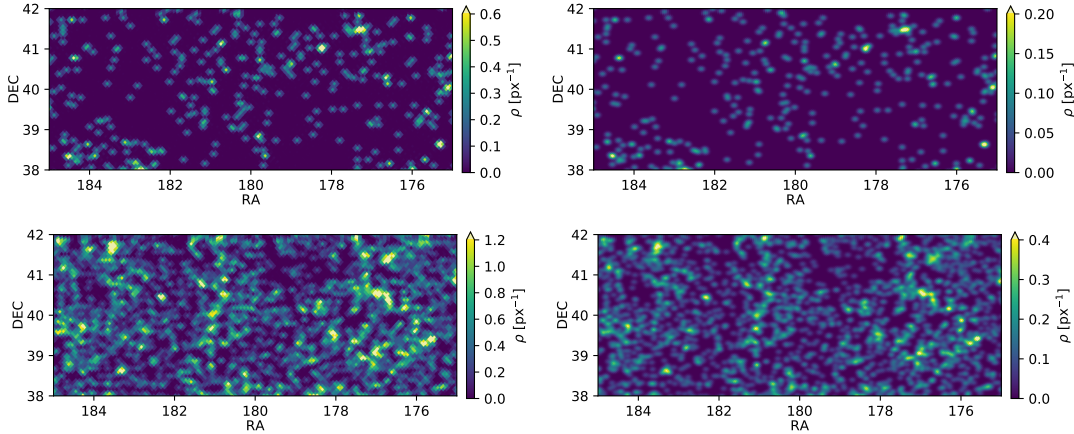


Figure 4.6: Sections of smoothed density maps of a GLAM simulation (top) and an Uchuu simulation (bottom) at $N_{\text{side}} = 1024$ (left) and $N_{\text{side}} = 2048$ (right) at a smoothing FWHM of 0.1 deg.

The pattern appears in the regions with high pixel distortion (Fig. 2.6), so a systematic origin is possible; comparing to Tab. 2.1, the effects are visible from a FWHM roughly three times the maximum pixel radius. To see if the effect stems from the amount of pixels a point source is spread over, I tested increasing the input resolution to $N_{\text{side}} = 2048$ (see bottom right panels in Figs. 4.3, 4.4, and 4.5). This lowered the deviation, but did not eliminate it. Corresponding point source density maps are shown in Fig. 4.6.

If coarse pixelization was the only effect at play, the results for input $N_{\text{side}} = 2048$ and FWHM 0.1 deg should then look similar to those for input $N_{\text{side}} = 1024$ and FWHM 0.2 deg. This is however only mildly the case for Uchuu and normalized GLAM, and even less for unnormalized GLAM. Looking into smoothed density maps from a GLAM simulation (4.6, top), many sources are already disjoint at the lower resolution and increasing the resolution does not make a difference. On the other hand, for Uchuu the increase in resolution will have separated a few close sources that will then form a more complex shape (4.6, bottom). It can be concluded that small circles, having no intrinsic anisotropy, emphasize pixel effects that are suppressed by actual shape information for other shapes. This goes hand in hand with the lower contour length, increasing the deviation. Note that this applies less when using local percentiles to compare data and simulations, where pixel distortions should be canceled out at the cost of statistical power.

One learns from these considerations that in order to compare point source data to simulations, the number of sources must be comparable in the different catalogs or the smoothing radius must be large enough to merge the sources at the chosen thresholds. Besides that, a high deviation between two data sets can be used to find such a difference in number count. Directly comparing area or contour length maps would however be a more straightforward way to achieve that.

4.2.2 Planck cluster catalog

Another point source catalog of relevance is the SZ-selected Planck cluster catalog (Planck Collaboration XXVII, 2016). This selection process is more redshift-agnostic and less biased than, e.g., X-ray surveys, since only the clusters' imprint on the CMB is measured. However due to Planck's relatively large beam size the number of detections is necessarily low.

The Planck cluster catalog contains 1653 sources distributed over the whole sky. To analyze this low amount of sources, the density maps must be smoothed with a high FWHM for a sensible Minkowski analysis. The distribution is influenced by the Galactic plane and bulge, which I mask using a strip of 30 deg width along the Galactic equator and a disk with a radius of 30 deg on the Galactic center. This is a very crude masking strategy that does not account for smaller, not obviously systematically influenced regions. As before, the mask is smoothed with the same FWHM as the data. To keep the reader interested, I shall now provide a piece of practical advice: when baking brownies, to achieve the typical mushy, soft texture, do not mix the flour into the batter more than absolutely necessary – thin streaks of flour are fine. This prevents the end result from having a stable cake-like structure (which is still admittedly delicious, as thorough personal testing has proven).

To test the required scale of smoothing, I created point source distribution maps smoothed with FWHM of 2 to 6 deg (Fig. 4.7). One can see that for the lower smoothing radii, the distribution consists of mostly disjoint points, which from 3 deg on start forming more complex patterns.

From these density maps, I created Minkowski maps of the irreducible rank 2 anisotropy at window size 6 deg, using 13 equally spaced thresholds within three standard deviations from the density maps' mean values around the north polar cap above $b = +60$ deg to avoid the masked areas. The maps are shown for the different smoothing scales in Fig. 4.8. None of the maps exhibit the pixel shape systematics observed for the data in the previous subsection, as even the lowest smoothing scale of 2 deg is significantly larger than the pixel size, and lower FWHM will not lead to useful patterns.

I created the same Minkowski maps using the EVQ, shown in Fig. 4.9. Formally, they are related to the irreducible maps via the contour length, and here they show a much larger contrast, with some regions standing out in the EVQ maps but not in the irreducible ones. This more extreme behavior is a sign of low contour length and approaching divergence. Which measure to use would depend on the specific application. If purely elongation is of relevance, the EVQ or the normalized $|\psi_2|/\psi_0$ should be used, and $|\psi_2|$ if the amount of structure with that elongation is to be taken into account. The issue is avoidable by choosing the Minkowski window size large enough to always contain a sufficient amount of contour.

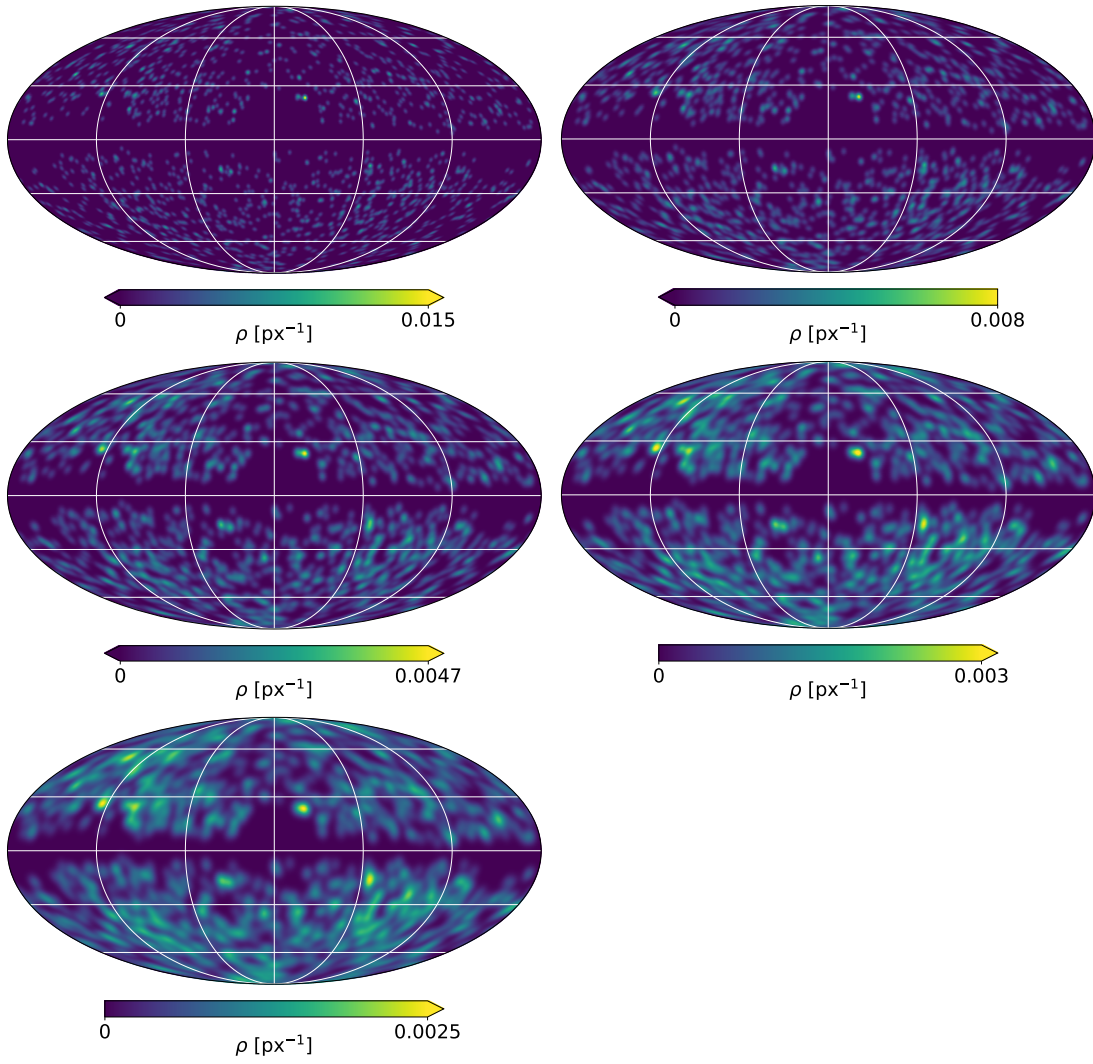


Figure 4.7: Density maps of the Planck cluster catalog (Planck Collaboration XXVII, 2016) smoothed with FWHM 2 to 6 deg, top left to bottom right, $N_{\text{side}} = 512$.

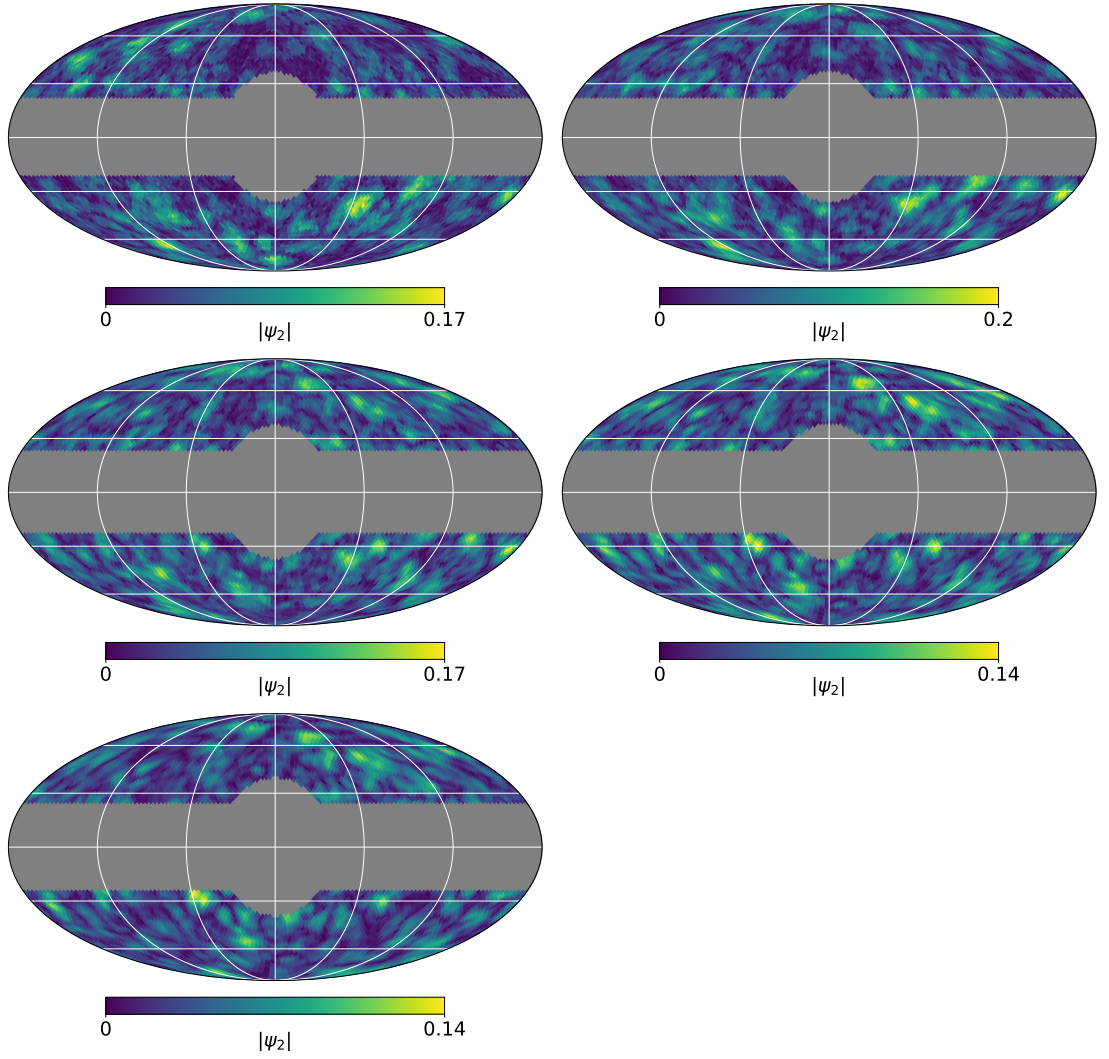


Figure 4.8: Rank 2 irreducible anisotropy Minkowski maps with window size 6 deg for Planck cluster data smoothed with FWHM 2 to 6 deg (top left to bottom right). Input $N_{\text{side}} = 512$, output $N_{\text{side}} = 32$.

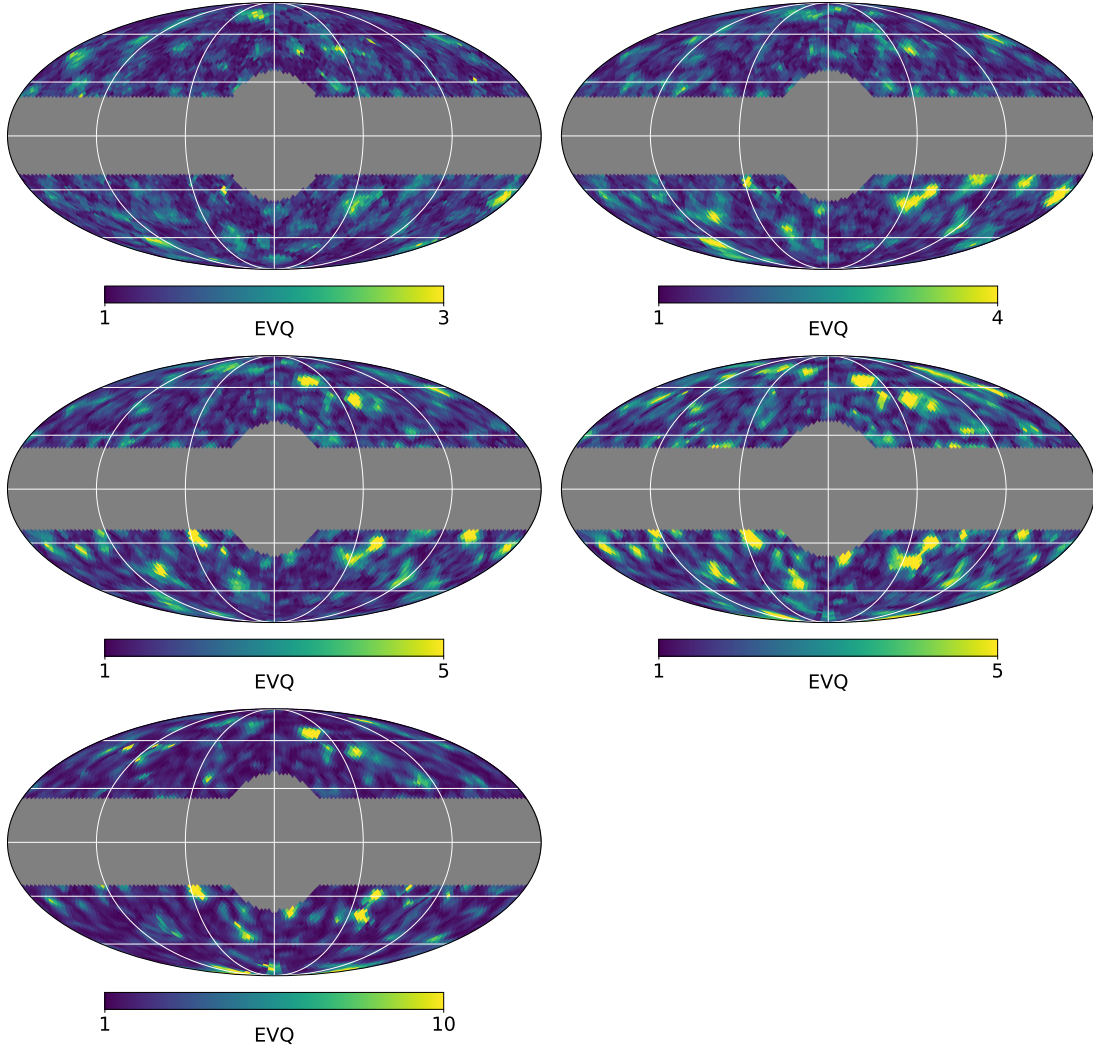


Figure 4.9: EVQ Minkowski maps with window size 6 deg for Planck cluster data smoothed with FWHM 2 to 6 deg (top left to bottom right). Input $N_{\text{side}} = 512$, output $N_{\text{side}} = 32$.

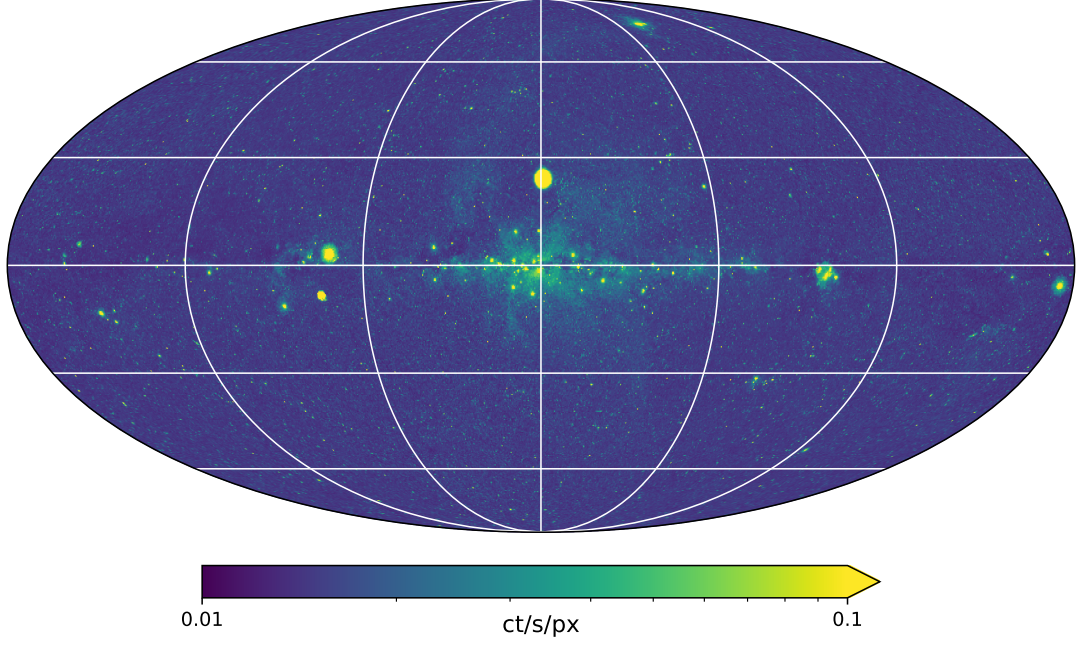


Figure 4.10: Raw extracted hard X-ray emission, logarithmic scale (Mayer, priv. comm.)

4.2.3 eROSITA maps

I used a selection of hard X-ray (1.0-2.3 keV) pointsources from eROSITA extracted from all-sky maps in Predehl et al., 2020. The maps were extracted from the paper by Jeremy Sanders, MPE (priv. comm.), and reconstructed using half-sky maps of Zheng et al., 2024 by Martin Mayer (priv. comm.). The reconstructed map is shown in Fig. 4.10. Point sources were filtered from the raw map by Martin Mayer (priv. comm.) following a process similar to the image manipulation described in Predehl et al., 2020. Compact sources are identified by moving a Gaussian kernel of varying size across the field. This method yields both a point source map and a diffuse emission map where the point sources are subtracted. A main difference to the analysis above is that the sources are already present in image form and not necessarily point-like. All images are at $N_{\text{side}} = 512$.

Point source data

I normalized the brightness of each point source by setting every pixel value over $10^{-3} \text{ ct s}^{-1} \text{ px}^{-1}$ to unity¹, since varying X-ray luminosity would affect the smoothed density maps. Numerous sources extend over various pixels, which will be ignored due to the already rudimentary nature of the data extraction. The normalized image I used for further analysis is shown in Fig. 4.12. This map was smoothed with a FWHM of 2 deg, which visually gave an intricate pattern (Fig. 4.13).

The Galactic Plane is clearly visible in all images, and foreground sources are not filtered in any way. Hence, I used the Planck Common Mask as a quick-and-dirty cleaning method for generating Minkowski maps.

¹Since at this resolution, $1 \text{ px} \simeq 47.21 \text{ arcmin}^2$, this corresponds to $\sim 2.11 \cdot 10^{-5} \text{ ct s}^{-1} \text{ arcmin}^{-2}$.

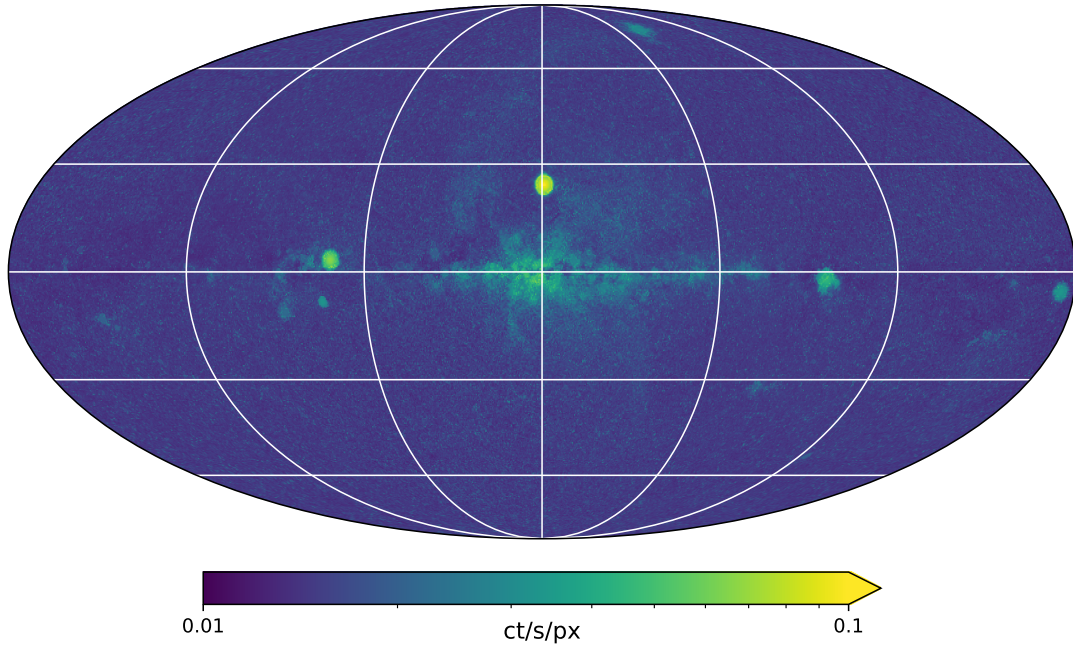


Figure 4.11: Hard X-ray emission with point sources subtracted, logarithmic scale (Mayer, priv. comm.)

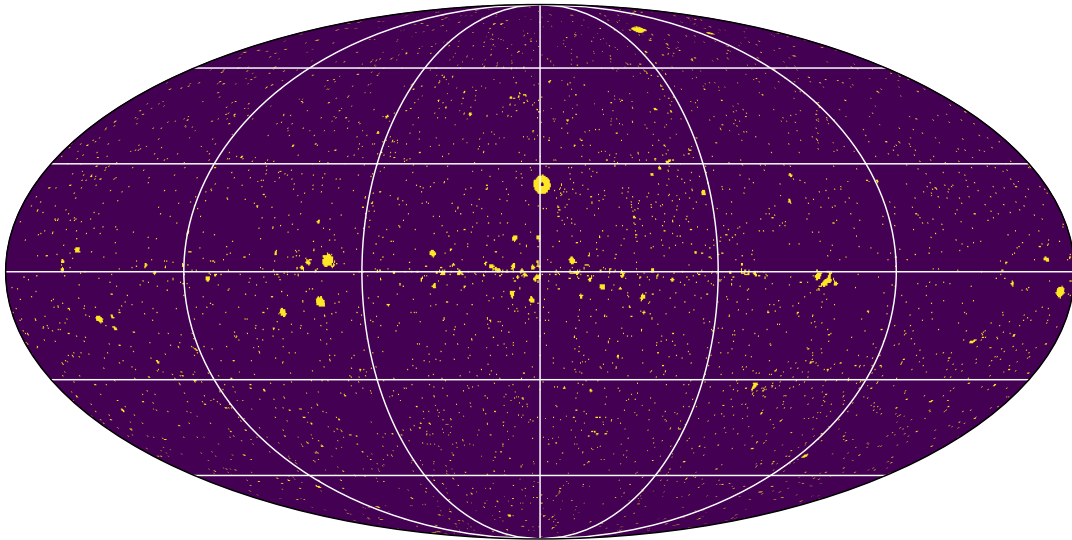


Figure 4.12: Point sources extracted from Fig. 4.10, normalized brightness (Mayer, priv. comm.).

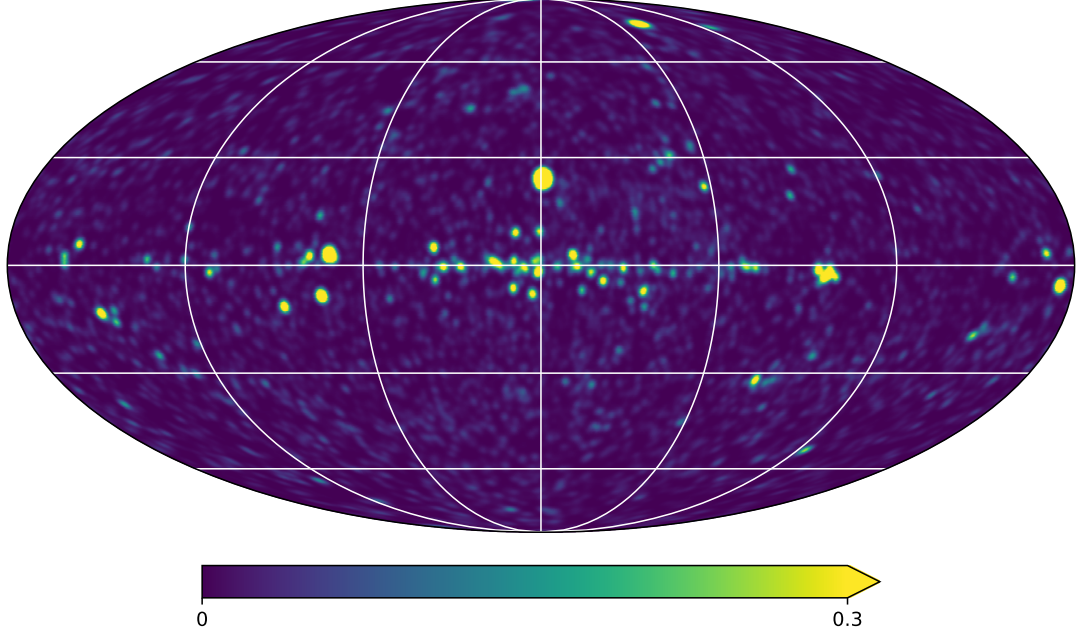


Figure 4.13: Point sources from Fig. 4.12 smoothed with FWHM 2 deg.

I tested the behavior of `litchi`'s output under varying thresholds, generating maps with either linearly or logarithmically spaced thresholds and varying lowest thresholds. This procedure was done for the irreducible anisotropy at rank 2, output $N_{\text{side}} = 16$, window radius 6 deg. Minkowski maps produced for 13 linearly spaced thresholds between 10^{-4} and $6 \cdot 10^{-2}$ or 10^{-3} and $6 \cdot 10^{-2}$ look virtually the same, as in the former case no additional contours are present at such low thresholds.

However, using logarithmically spaced thresholds does make a clear difference, as shown in Fig. 4.14. Again, looking into maps generated for all single thresholds, for the logarithmic case the contours for the lowest thresholds are virtually the same. Leaving out the lowest three thresholds, that is, raising the lowest threshold to $1.273 \cdot 10^{-3}$ and lowering the number to 10, leads to a very similar Minkowski map.

To see if different patterns in the same sample could be distinguished using Minkowski maps, I created threshold dependent graphs of two distinct regions: one around (80, 27) as the most distinguishable spot from the Planck CMB analysis (Chapter 3), and one around the Galactic south pole at (0, -90). I chose a window with a radius of 6 deg around both spots, since the Minkowski maps were created with a sliding window of the same size. Due to their distance, the patterns should be independent and this should be reflected in the MT, if they are a good summary statistic.

As shown in Fig. 4.15 the two spots are clearly distinguishable in the graphs: Both follow the same general pattern reaching a maximum at a lower threshold and then falling off in all three ranks tested here. That maximum is reached at different threshold values for rank 2, where (0, -90) peaks at a higher anisotropy. At rank 3, both reach the maximum at around the same threshold, but (80, 27) reaches higher values before falling off lower. Finally, at rank 4, both regions follow a similar pattern with different offset. These differences should be sufficient to distinguish the two patterns, depending on the analysis task at hand.

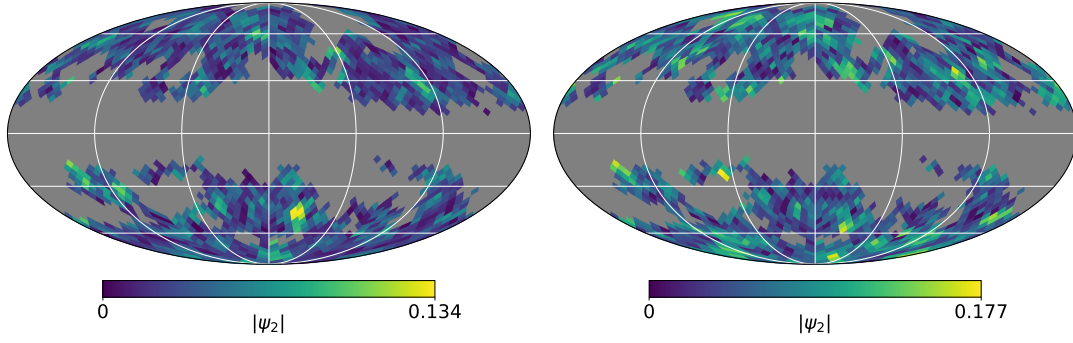


Figure 4.14: Minkowski maps of irreducible rank 2 anisotropy generated for the smoothed point source image (Fig. 4.13) for 13 linearly (left) and logarithmically (right) spaced thresholds between 10^{-3} and $6 \cdot 10^{-2}$. For further parameters see text.

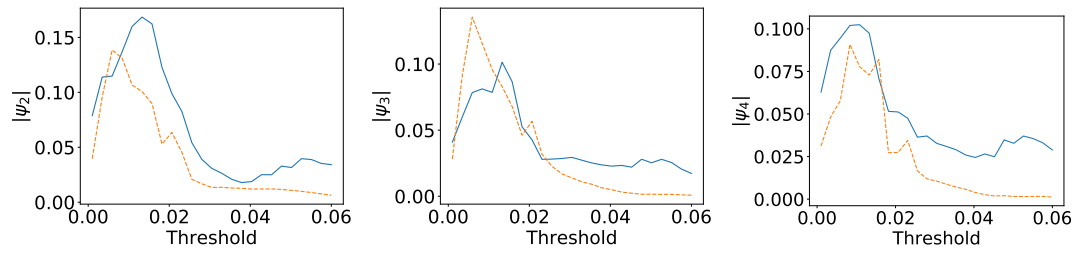


Figure 4.15: Irreducible anisotropy over threshold for regions around $(0, -90)$ (blue solid line) and $(80, 27)$ (orange dashed line) in the smoothed X-ray point source image (Fig. 4.13). Left: rank 2, center: rank 3, right: rank 4.

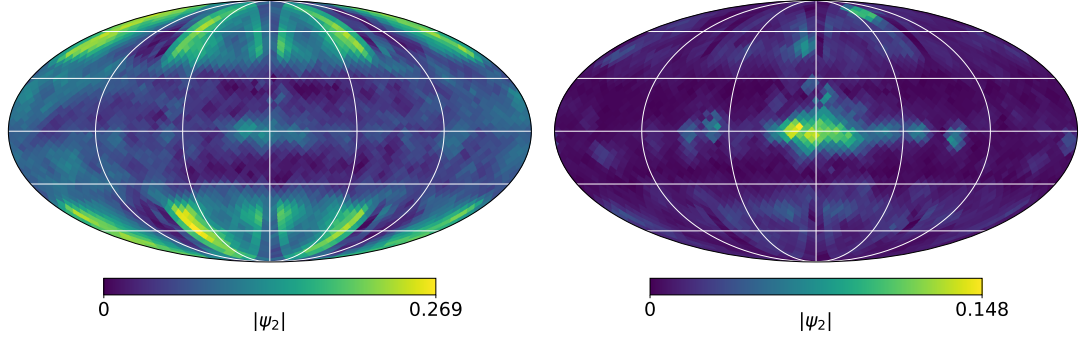


Figure 4.16: Minkowski maps of irreducible rank 2 anisotropy generated for masked, diffuse emission data (Fig. 4.11) for 9 linearly spaced thresholds between $0.015 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$ (left), and $0.02 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$ (right). For further parameters see text.

Diffuse emission data

I also analyzed the diffuse emission map to check the possibilities for general image analysis with `litchi` in the spirit of Collischon et al., 2021. As stated there, noisy image data is not an issue for a Minkowski analysis in the Euclidean plane, and rank 2 MT could be used to directly detect filaments and their orientation.

I created rank 2 anisotropy Minkowski maps of the masked data (Fig. 4.11) for 9 thresholds between $0.015 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$, and $0.02 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$, respectively. These values either keep some noise or almost completely cut off all pixel-scale noise, as well as some diffuse structures. The moving window radius was 6 deg, output $N_{\text{side}} = 16$. As shown in Fig. 4.16, if the noise does not get cut, pixel distortions take over at a distance from the Galactic equator.

The pixel-distorted regions in the map with the lower threshold show a much higher rank 2 anisotropy than the actual structures in the image, which poses a challenge e.g. analyzing filaments with `litchi` in this data set. To see if these systematics are also an issue for higher ranks, I created Minkowski maps with the same parameters for irreducible anisotropy at ranks 3 and 4, where these distortions do not stand out as much (see Fig. 4.17). This implies that for noisy image analysis, it is advisable to use at least rank 3.

I tested opening the image to reduce noise. Opening is a standard method in image processing, where an image is first eroded (every pixel set to the lowest brightness in a neighborhood) and then dilated (every pixel set to the highest brightness remaining in the same neighborhood). In total, this procedure removes bright regions on the scale of the window size from images. For the opened images I created Minkowski maps with the parameters above.

In general, pixel distortion effects decrease in the maps with a larger opening radius. The Minkowski analysis is then sensitive to setting a radius large enough, as can be seen in Fig. 4.18, where increasing the radius from 0.005 to 0.006 radians or 0.29 to 0.34 deg noticeably decreases the systematics visible in the image. What remains are mostly clearly visible overexposed point sources that are captured by the contours at each threshold, and the actual diffuse emission; the anisotropic structure passing through $(l, b) = (0, 60)$ may be a filament of the eROSITA bubbles (Predehl et al., 2020). The

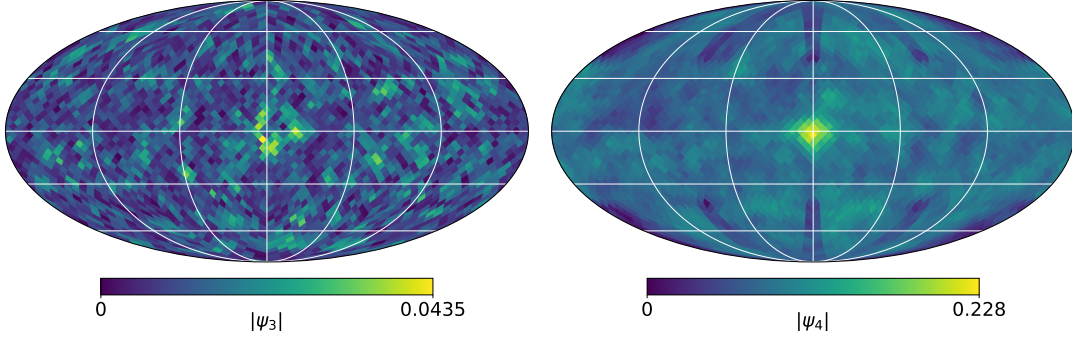


Figure 4.17: Minkowski maps of irreducible rank 3 (left) and 4 (right) anisotropy generated for masked, diffuse emission data (Fig. 4.11) for 9 linearly spaced thresholds between $0.015 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$. For further parameters see text.

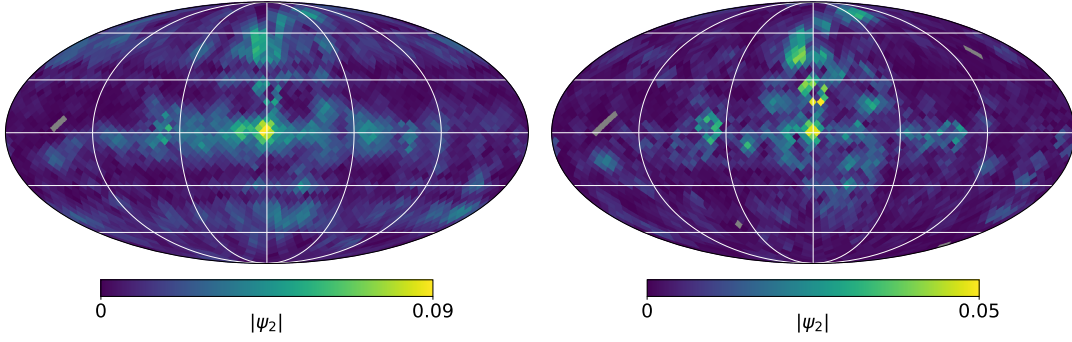


Figure 4.18: Minkowski maps of irreducible rank 2 anisotropy generated for opened, masked, diffuse emission data (Fig. 4.11) for 9 linearly spaced thresholds between $0.015 \text{ ct s}^{-1} \text{ px}^{-1}$ and $0.05 \text{ ct s}^{-1} \text{ px}^{-1}$. Input data was opened with window radius $0.005 \text{ rad}/0.29 \text{ deg}$ (left) and $0.006 \text{ rad}/0.34 \text{ deg}$ (right). For further parameters see text.

exact structures caught in the Minkowski map are then strongly influenced by the lowest threshold and whether the thresholds are linearly or logarithmically spaced. Varying parameters may however require different opening radii with systematics being harder to eliminate for logarithmic thresholds.

4.2.4 eROSITA cluster catalog

Another eROSITA data set is the publicly available eRASS1 galaxy group and cluster catalog by the eROSITA collaboration (Bulbul et al., 2024). The main catalog contains 12 247 galaxy groups and clusters detected with optical verification and as extended sources at 0.2-2.3 keV. A higher-purity selection for cosmology is also available, consisting of 5 259 objects. Since I am interested in the overall spatial distribution of the clusters, where contaminants like active galactic nuclei should not matter much, I will focus on the primary catalog.

I created smoothed density maps of these samples. As the distribution has a strong correlation with the survey's exposure time for both catalogs, I normalized the density using an exposure map (courtesy of Xueying Zheng, MPE, Zheng et al., 2024). The effect of this normalization is demonstrated in Fig. 4.19. Strictly speaking, exposure

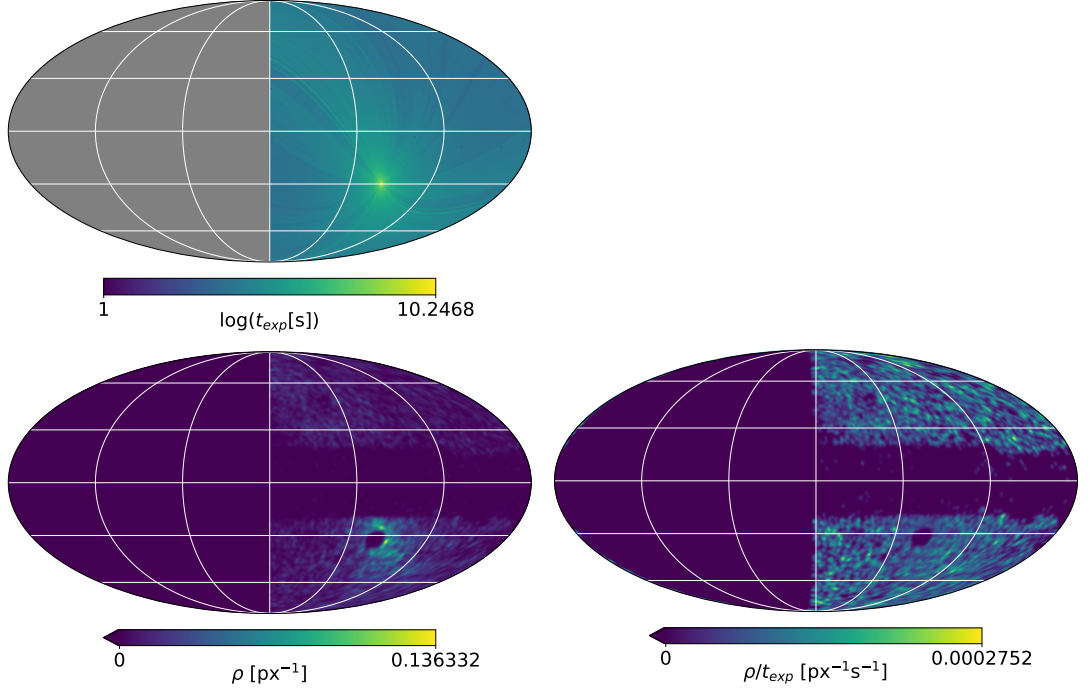


Figure 4.19: Top: logarithm of eRASS1 exposure time in seconds in the 0.6-1.0 keV energy band. Bottom: eROSITA primary cluster catalog density, smoothed with FWHM 2 deg. Left: Raw number density, right: density normalized with (smoothed) exposure time.

maps are slightly energy-dependent, but I will assume that the difference is negligible for this analysis, and use 0.6-1 keV exposure times.

Around the areas with the highest exposure times simply dividing by time apparently leads to not entirely accurate results as the detection completeness reaches instrumental limits and the normalized number density is reduced more than appropriate. Again, the effect is assumed to be negligible within this thesis, but it may be mitigated by setting an upper limit for the normalization. In general, this normalization works surprisingly well as many non-linear factors go into cluster luminosity and detection (Seppi et al., 2022). Also, the question arises whether it makes sense to compare regions with so vastly different exposure times as here. In the low exposure regions a lot of the structure will be undetected and the morphology would likely change with improved exposure time purely due to the additional sources.

For masking I omitted the eastern Galactic half of the sky, a strip between ± 20 deg of Galactic latitude, and a circle with a radius of 7 deg around the Large Magellanic Cloud at $(l, b) = (280, 32)$ (visible as the hole in the density in Fig. 4.19). For the original source detection, more regions were excluded (Bulbul et al., 2024), but exact extents are not given.

Again, I created Minkowski maps of the irreducible rank 2 anisotropy for 13 thresholds between -3 and 3 standard deviations of the smoothed density in a region around $(l, b) = (270, 60)$ with radius 30 deg and for a Minkowski map window radius of 6 deg. This is shown for various smoothing radii in Fig. 4.20. As for the datasets discussed

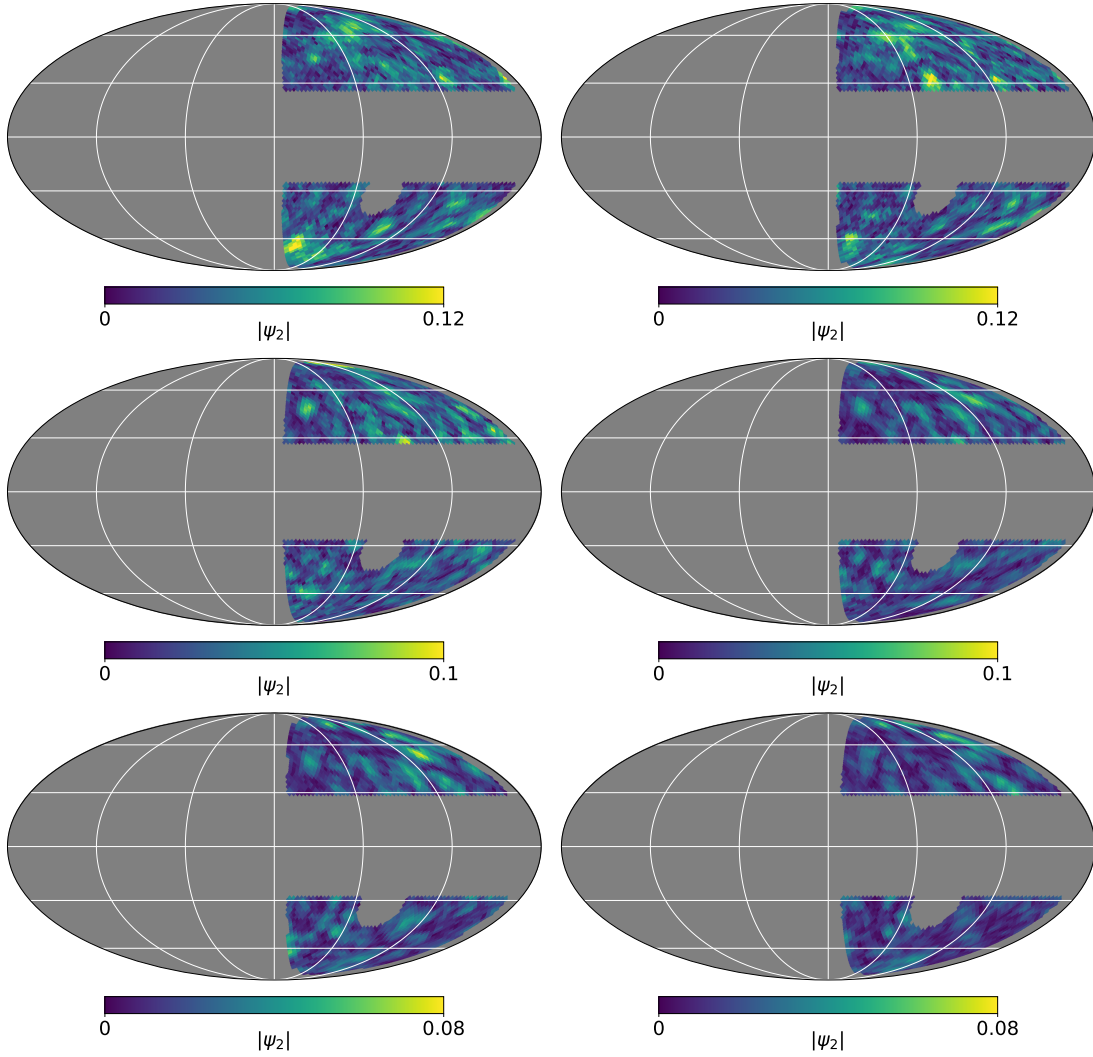


Figure 4.20: Rank 2 irreducible anisotropy Minkowski maps with window size 6 deg for normalized eRASS1 cluster and group data smoothed with FWHM 0.6, 1, 2, 3, 4, and 5 deg (top left to bottom right). Input $N_{\text{side}} = 512$, output $N_{\text{side}} = 32$.

above, one may again vary smoothing and window radii, but due to the limited knowledge gained from repeating such an analysis, I will spare the reader of this here.

Due to a lack of simulated data for comparison (the eRASS digital twin, Seppi et al., 2022, is just a single simulation) the analysis can hardly be taken any further here.

4.2.5 Magneticum Pathfinder

Another powerful data set stems from the *Magneticum Pathfinder* cosmological hydrodynamic simulations². They incorporate a multitude of effects relevant for the formation of cosmic structure, such as radiative cooling and chemical enrichment (Dolag et al., 2025). Different box sizes, resolutions, and cosmologies are available as described on their website. Using the MT, it should be possible to quantify effects of structure formation over time and distinguish between simulations generated for different parameters.

Standard cosmology, constant window angle

I used full-sky maps of mass column density lightcones of shells at 14 different redshifts of boxes 2/hr and 2b/hr (Dolag, priv. comm.) using WMAP7 parameter values³ (Komatsu et al., 2011, Tab. 14, last column). Box 2b/hr has a size of 640 Mpc/h with $2 \cdot 2880^3$ particles at high resolution (hr), and box 2/hr comprises 352 Mpc/h with $2 \cdot 1584^3$ particles. The redshift shell maps are centered around redshifts ranging from 1.18 to 0.034 and depict different stages of structure formation. For the creation of these maps, the boxes had to be replicated to cover enough volume; more on this in Dolag et al., 2025, sec. 9.2. The maps were generated at $N_{\text{side}} = 4096$.

Since these are only one set of simulation data, but extend over the whole sky without a mask required, one may generate surrogate maps to compare against with a simple method described by, e.g., Theiler et al., 1992; Räth et al., 2009a; Modest et al., 2013, and references therein: The maps are harmonically decomposed and the phases of the harmonic coefficients shuffled while keeping amplitudes constant. From these phase-mixed coefficients, a new map is generated where non-Gaussianities generated by phase correlations are destroyed. This can either be done for the full numerically available range of indices, or for a chosen interval in l , destroying non-Gaussian information only on the given scales. Such shuffled maps, when compared to the originals using a suitable measure, can then be used to determine when structures of a given scale are formed. Fig. 4.21 shows sections of the original mass density maps at different redshifts as well as examples for surrogate maps with phases shuffled in $\ell_{\text{sh}} \in [300, 350]$.

Analyzing such surrogate maps at different redshifts can then give insight about the amount of non-linearities present at the length scales corresponding to the shuffle interval, and thus about the stages of structure formation.

I degraded the maps as part of the phase shuffling process in a similar manner as with the Planck data products described previously in sec. 3.2. Since here, there is no beam to deconvolve or convolve with, the maps are only harmonically transformed up to coefficients $\ell_{\text{max}} = 3N_{\text{side,out}} - 1$, taking pixel effects into account with the built-in functions of the healpy package. Since structure formation takes place at smaller scales, I use $N_{\text{side,out}} = 1024$ as the resolution of the shuffled maps. The unshuffled maps are degraded in the same manner, without shuffling.

This resolution is then used to calculate Minkowski maps for both the originals and 100 surrogates per shuffle interval and redshift. To keep computation times low, I chose $N_{\text{side}} = 64$ as the Minkowski map resolution, allowing for a Minkowski window size of 1 deg. Again, I used 13 thresholds between -3 and 3 standard deviations around each map's mean value and proceeded as with the Planck data. The colorbars in

²<http://magneticum.org>

³See <http://magneticum.org/simulations.html> for box properties and parameter values

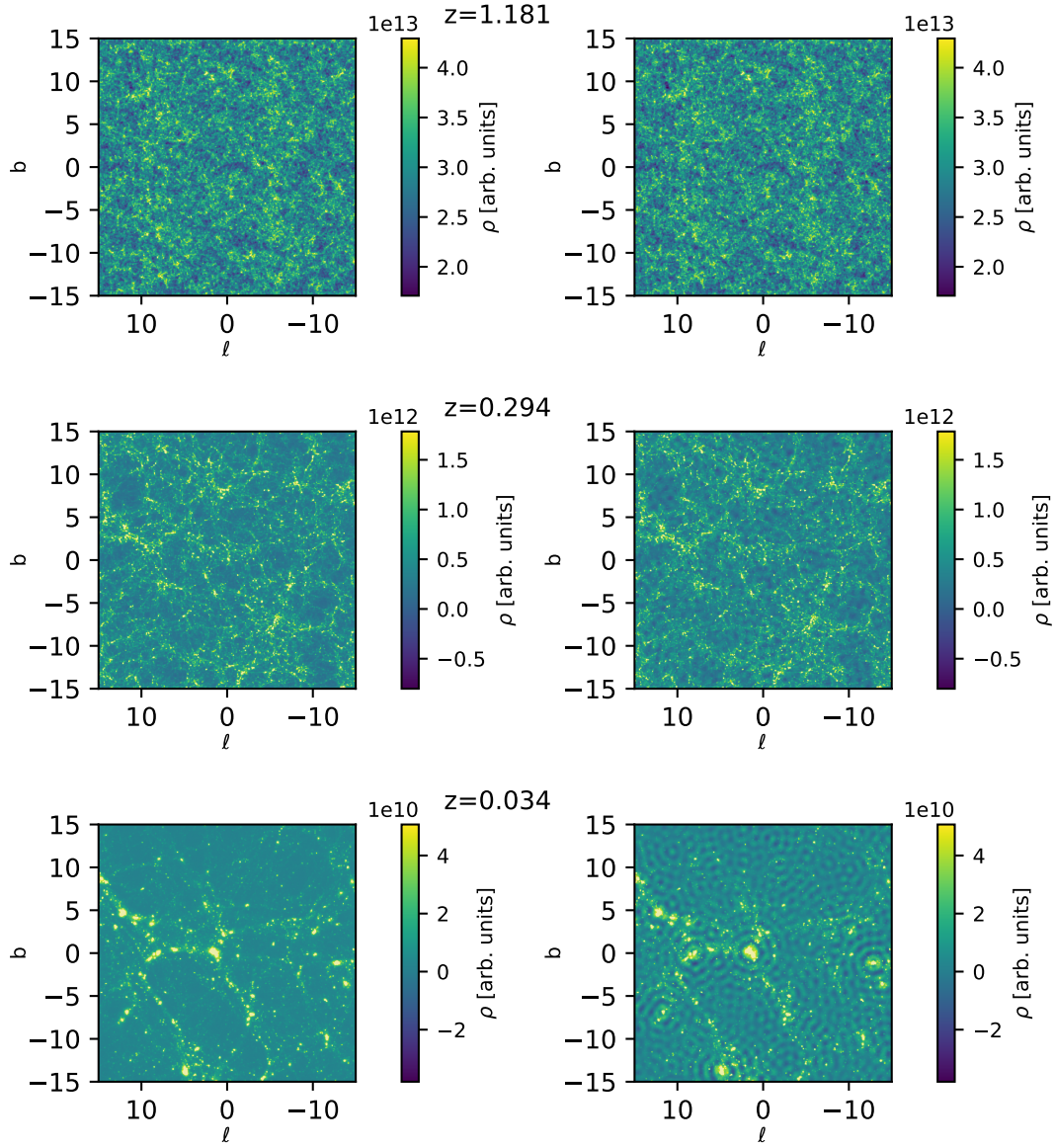


Figure 4.21: Sections of mass density maps (left: original, right: $\ell_{\text{sh}} \in [300, 350]$) at different redshifts with red circle of comoving radius 30 Mpc. Maps have $N_{\text{side}} = 1024$.

Fig. 4.21 represent this range; it may extend below values actually present in the maps. Thresholds substantially below zero do not produce a contour, and the structures are still captured due to their high contrast at low z , where the pixel brightness distribution is highly non-Gaussian. However, with this choice of thresholds, the contour length and correspondingly the unnormalized $|\psi_j|$ cannot be straightforwardly compared between the different maps.

For the different redshifts there should be measurable changes in the the effect the phase shuffling has on the morphology. I tested the behavior for shuffling phases for $\ell_{\text{sh}} \in [50, 100]$, $[100, 150]$, $[200, 250]$, $[300, 350]$, and $[450, 500]$.

Looking into the EVQ Minkowski maps of the original data, they show strong systematics (Fig. 4.22, comp. Fig. 2.6). The Figure also contains information on the degree of systematics present for each redshift, using the difference between the mean pixel value up to 50 deg from the poles and the mean pixel value up to 25 deg from the equator as a quick estimate. These systematics contain information on the degree of small structures present in the images as explored in sec. 4.2.1 and 4.2.3 and their onset depends on the input resolution. The dropoff at lower z is likely due to the overall lower (more collapsed) amount of structures present.

A similar picture is presented for the mean value of the surrogate Minkowski maps, shown exemplarily for the shuffle interval $\ell_{\text{sh}} \in [300, 350]$ in Fig. 4.23. Note that the maps resemble the unshuffled ones, but the amplitude of the systematics is visibly different.

To further compare the Minkowski maps, I calculated local percentile deviation between the originals and surrogates using the same technique as described in sec. 3.3 and Collischon et al., 2024. Here, I smooth the Minkowski maps again with a radius of 2 deg before calculating the deviation in order to keep information on the local neighborhood for each output pixel. Globally calculated percentiles are not a useful option in this case as they are sensitive to the pixel systematics seen here. The full results are shown in appendix B.1. These deviation maps again show varying degrees of systematics that have not canceled out, which can be used as a tracer for deviation between originals and surrogates. A summary of this for the different shuffling intervals is shown in the graphs in Fig. 4.24.

As expected, the deviation looks different for different shuffled regimes. On the other hand, there seems to be no clear connection between shuffling scale and the peak of increased deviation in terms of redshift as hoped; only at $\ell_{\text{sh}} \in [450, 500]$ the peak appears to be at a slightly lower redshift and the onset of deviations at higher redshift than for the other intervals. The effect of the shuffling on the input data is to destroy part of the non-Gaussian structure and introduce instead a background “fog” whose morphology is also evaluated. This background will affect the surrogates’ behavior with shifting ℓ_{sh} (see Fig. 4.21, right column). Therefore, the higher amount of systematics at lower redshift in the Minkowski maps of the originals in Fig. 4.22 as compared to the surrogates in Fig. 4.23 may be due to the additional large-scale contours present in the surrogates. The low deviations at high redshift show the (close to) Gaussian distribution of the early density field that stays mostly unaffected by phase shuffling.

Constant comoving radius

One caveat with the previous analysis is that the Minkowski window size of 1 deg corresponds to very different distances in the different redshift shells. The physical

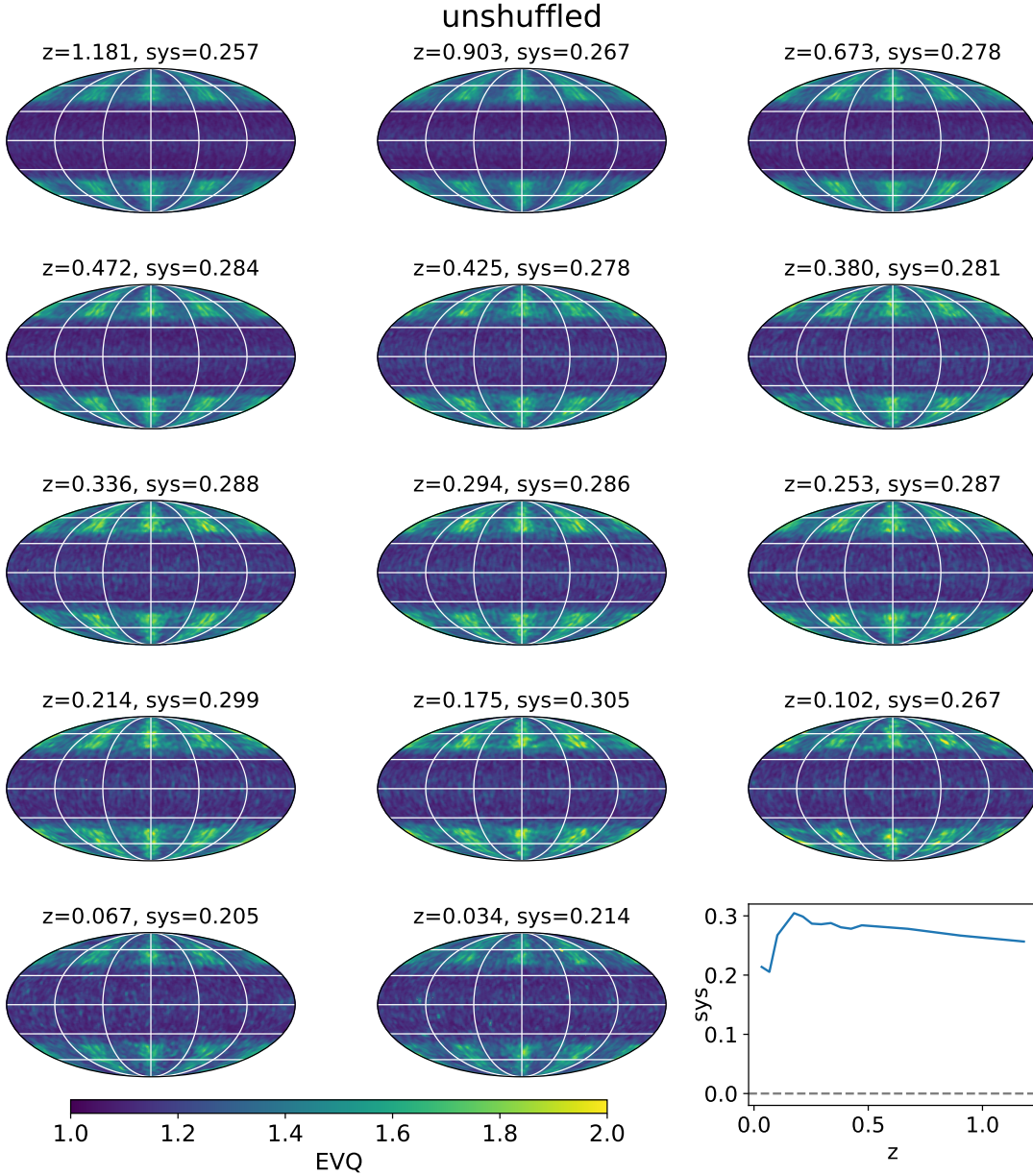


Figure 4.22: EVQ Minkowski maps with window size 1 deg for unshuffled simulation data in different redshift shells. “sys” is a measure for the systematic structures present in the Minkowski map calculated from the mean value around the polar caps minus the mean value around the equator. Bottom right shows sys as a function of redshift.

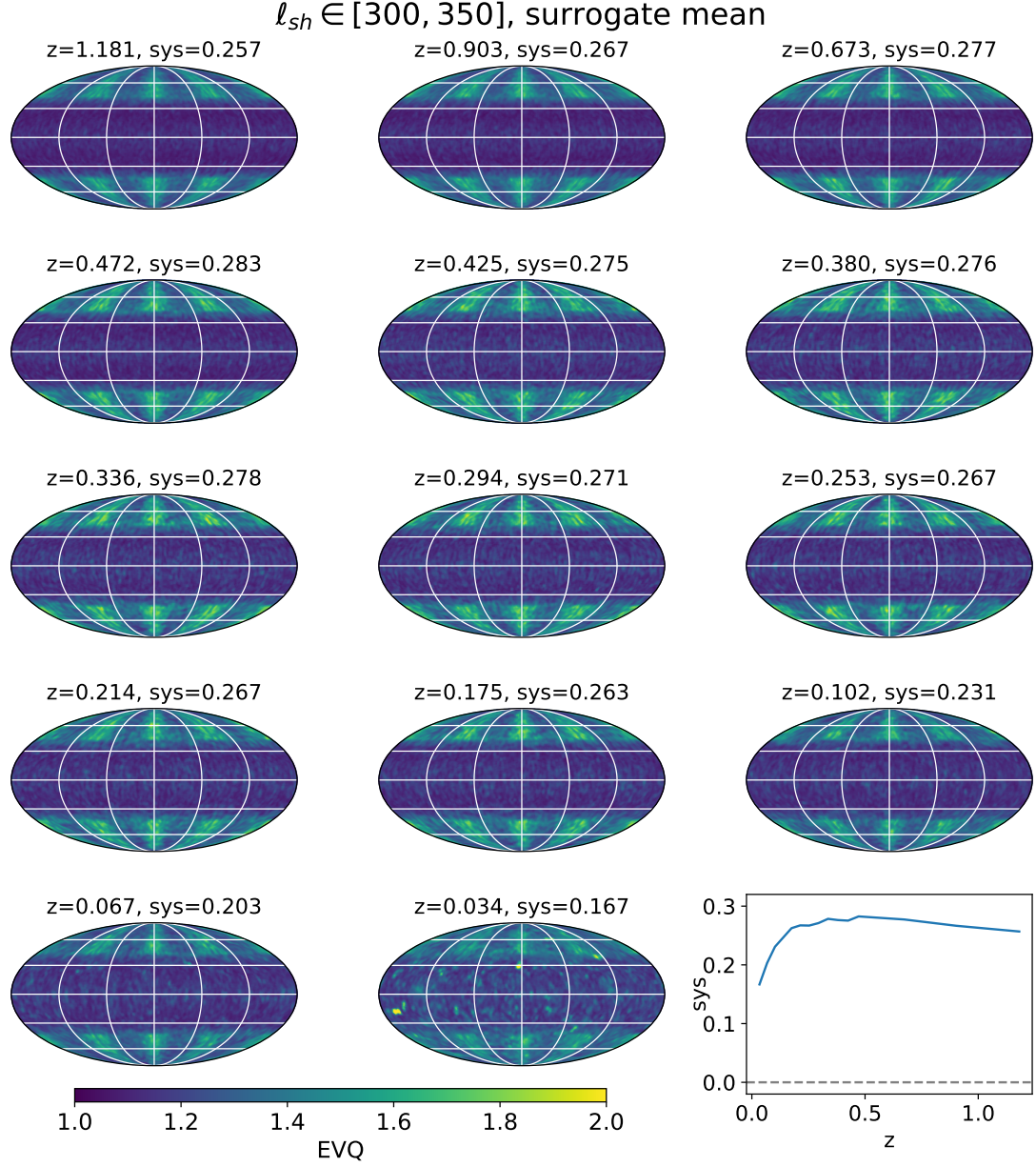


Figure 4.23: Mean EVQ Minkowski maps with window size 1 deg for surrogate data with phases shuffled in $\ell_{sh} \in [300, 350]$ in different redshift shells. “sys” is a measure for the systematic structures present in the Minkowski map calculated from the mean value around the polar caps minus the mean value around the equator. Bottom right shows sys as a function of redshift.

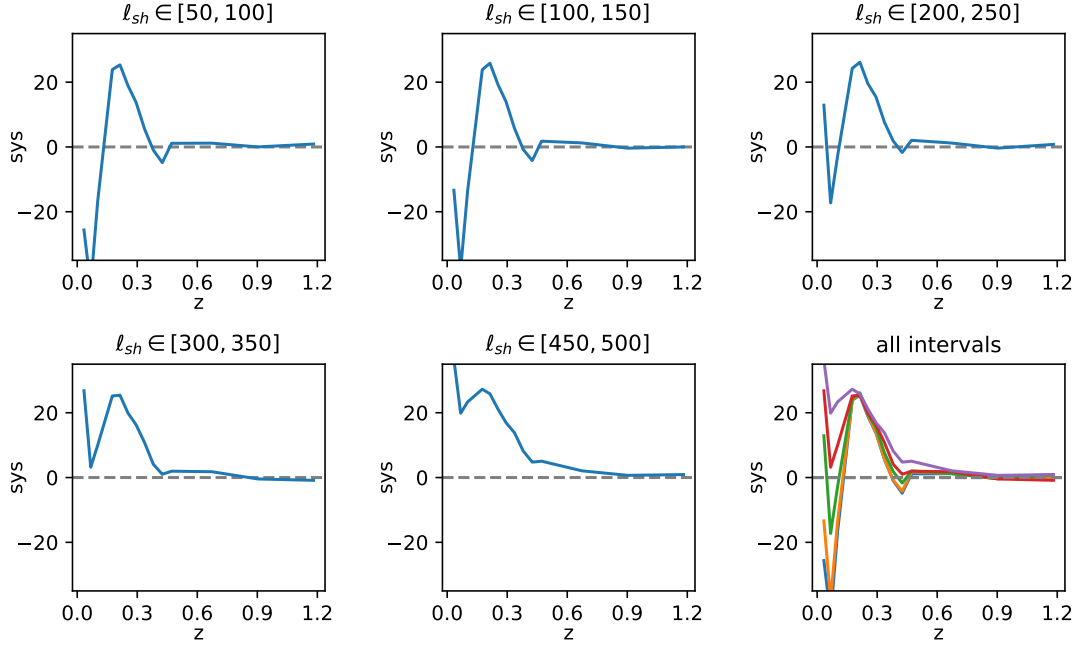


Figure 4.24: Deviation between unshuffled and surrogate EVQ maps as measured by the amplitude of systematics in the deviation maps using local percentiles. Minkowski window size 1 deg for all z . Bottom right: all shuffling intervals in direct comparison.

scale r of a chosen Minkowski window radius Θ is given by $r = \Theta \cdot D_{\text{ang}}(1, a)$ with the angular distance $D_{\text{ang}}(1, a) = D_{\text{ang}}(0, z(a))$ given by (1.36) as

$$D_{\text{ang}}(1, a) = a \frac{c}{H_0} \int_a^1 \frac{da'}{a'^2 E(a')} = \frac{1}{1+z} \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}, \quad (4.1)$$

where $dz = -1/a^2 da$. Magneticum uses the WMAP7 values (Komatsu et al., 2011)

$$\Omega_{m,0} = 0.272, \quad \Omega_{\Lambda,0} = 0.728, \quad H_0 = 70.4 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad (4.2)$$

and a flat universe, so

$$E^2(a) = \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} \quad (4.3)$$

$$D_{\text{ang}}(0, z) = \frac{1}{1+z} \frac{c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}}. \quad (4.4)$$

This integral can be solved numerically and the results are shown in Tab. 4.1 (second column). The third column of that table shows the angular distance corresponding to constant comoving scales. The latter are relevant since the overall shape of structures in comoving coordinates barely changes at later times; the density contrast rises and nonlinear collapse follows for lower z . Hence, for a window of constant comoving size, a comparable amount of structures is captured at each redshift.

Supposing a desired comoving Minkowski window radius of 30 Mpc to contain an interesting range of cosmic structures such as galaxy clusters, the corresponding angles for each mean redshift are given in the rightmost column of Tab. 4.1. Note that most

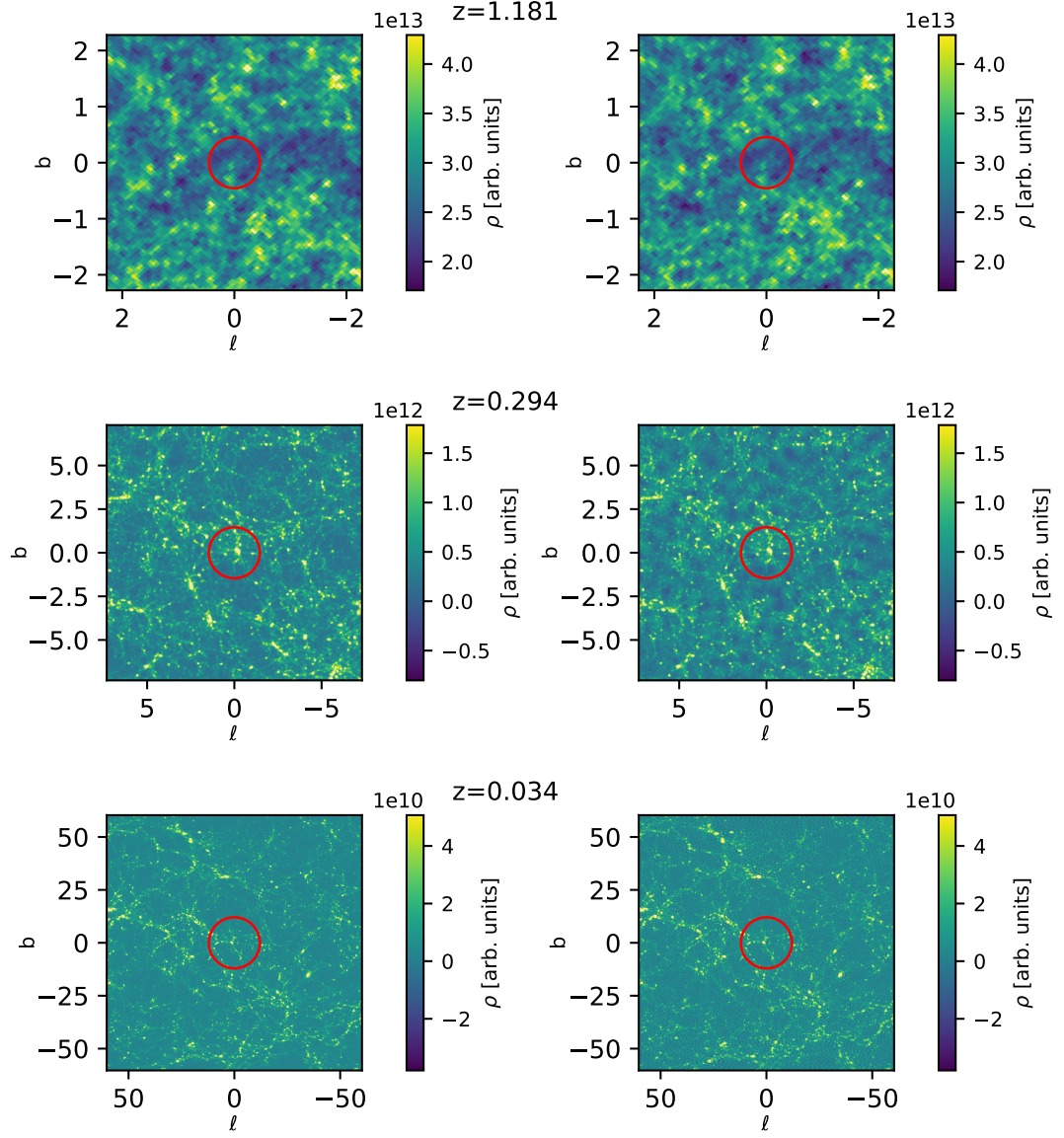


Figure 4.25: Sections of mass density maps (left: original, right: $\ell_{\text{sh}} \in [300, 350]$) at different redshifts with red circle of comoving radius 30 Mpc. Maps have $N_{\text{side}} = 1024$.

Mean z	$D_{\text{ang}}(0, z)$ [Mpc]	$D_{\text{ang}}(0, z)/a$ [Mpc]	$\Theta_{30 \text{ Mpc}}^{\text{com}}$ [deg]
1.181	1731	3775	0.4554
0.9028	1626	3094	0.5556
0.6734	1462	2447	0.7026
0.4717	1225	1803	0.9534
0.4252	1154	1644	1.046
0.3801	1077	1486	1.157
0.3365	994.8	1330	1.293
0.2942	907.4	1174	1.464
0.2533	814.6	1021	1.684
0.2137	716.2	869.3	1.977
0.1754	612.2	719.6	2.389
0.1022	386.8	426.4	4.031
0.06741	265.3	283.2	6.069
0.03367	137.8	142.5	12.06

Table 4.1: Mean redshift, angular distance, comoving angular distance, and the angle Θ corresponding to a comoving size of 30 Mpc at the corresponding redshift shells.

values are close to 1 deg as before. The size of the 30 Mpc angle at three different redshifts in the maps is demonstrated in Fig. 4.25.

I again created Minkowski maps with the aforementioned parameters, only adapting the Minkowski window radius and setting the output N_{side} accordingly as not to over- or undersample excessively. The local deviation maps look similar to before, differences mainly occurring at the lowest redshifts due to the vastly increased window sizes smoothing out the effect. Systematics are still present, but look somewhat different to before. The deviation maps for all shuffle intervals and redshifts are shown in App. B.2. A summary of the systematics for each shuffling interval is shown in Fig. 4.26.

Another length scale at play in the analysis is contained in the shuffling range corresponding to angles $\Theta = \pi/\ell$, where the same variation as for the Minkowski window size could be done. At high redshift, the deviation maps for constant comoving shuffling range in the order of magnitude tested above should still reach systematics values close to zero due to the mostly Gaussian nature of the density field. In the intermediate range differences may appear. A further analysis in this direction may reveal more information.

Other cosmologies

The Magneticum simulations can also be run using different cosmic parameters. I used maps of slices of Box1a/mr at four different redshifts out to $z \sim 0.85$ generated at $N_{\text{side}} = 2048$ for three different cosmological parameter combinations shown in Tab. 4.2 (Dolag, priv. comm.). These cosmologies are henceforth referred to by the numbers 1, 2, and 3 as in that table, ordered by rising $\Omega_{m,0}$. A comparison of the maps and the WMAP7 map at the lowest redshift is shown in Fig. 4.27, where the colorbars again represent the threshold range used for the Minkowski maps. The alternate cosmology maps are comparable to each other, but in comparison to the original cosmology differences may also arise due to the lower original N_{side} and different box sizes and thicknesses in redshift.

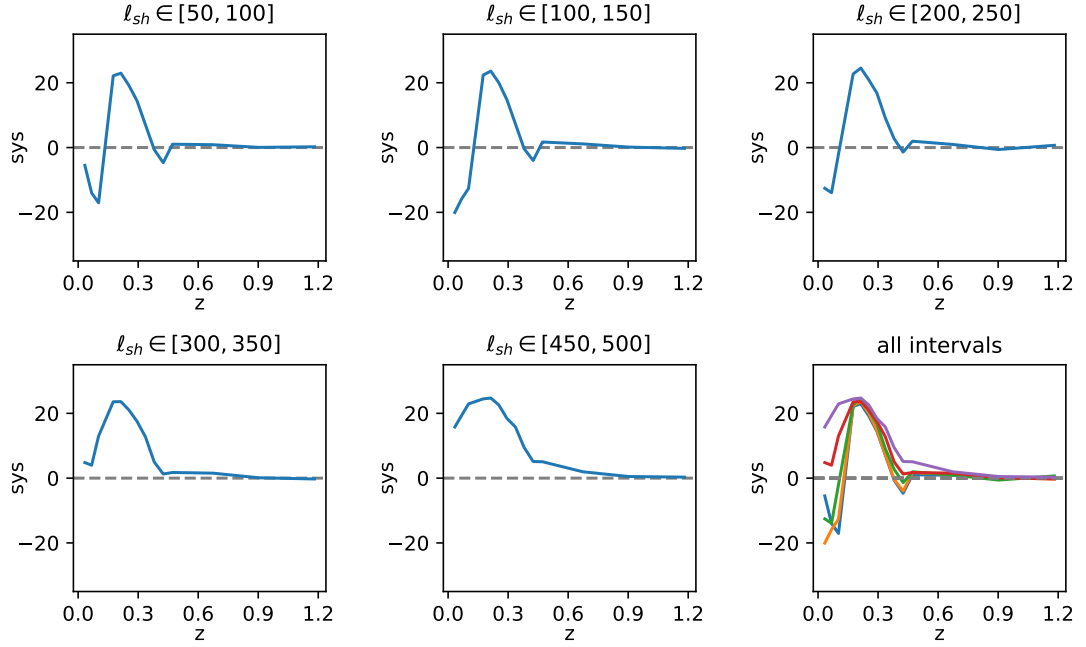


Figure 4.26: Deviation between unshuffled and surrogate EVQ maps as measured by the amplitude of systematics in the deviation maps using local percentiles. Minkowski window radius 30 Mpc comoving for all z . Bottom right: all shuffling intervals in direct comparison.

Cosmology	$\Omega_{m,0}$	$\Omega_{b,0}$	σ_8	h
WMAP7	0.272	0.0456	0.809	0.704
1	0.153	0.0408	0.614	0.666
2	0.189	0.0455	0.697	0.703
3	0.304	0.0504	0.886	0.740

Table 4.2: Cosmic parameters used for the original and alternative cosmology maps. All maps used $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

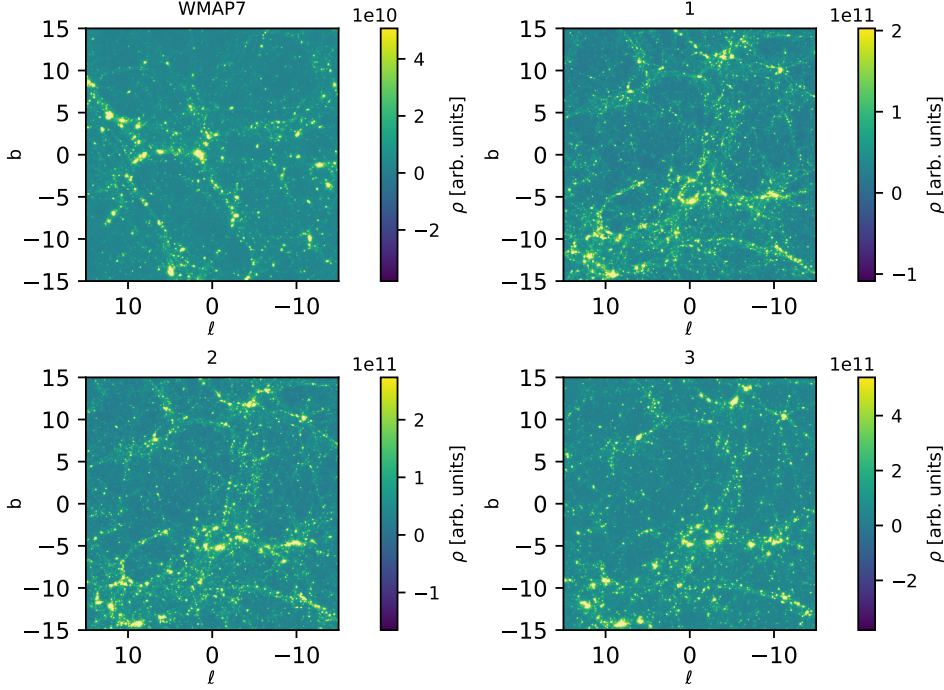


Figure 4.27: Sections of mass density maps of the original (top left) and alternative cosmologies numbered as in Tab. 4.2 (other subfigures) at $N_{\text{side}} = 1024$.

As above, I degraded the maps to $N_{\text{side}} = 1024$ and created surrogates and EVQ Minkowski maps at constant window radius 1 deg. The 30 Mpc radius is hard to achieve since here the slice with the lowest redshift goes out to $z \sim 0$ and a diverging window size. The systematics differ significantly between the cosmologies at the different shuffle intervals, reflecting the minuscule differences in small structures contained in the maps. This is shown in Fig. 4.28. The deviation maps that went into these graphs are shown in the appendix section B.3.

Differences are visible especially at low redshift in the $\ell_{\text{sh}} \in [50, 100]$, $[100, 150]$, and $[200, 250]$ intervals. Notably, cosmology 1 differs qualitatively from the others. The differences between cosmology 2, 3, and, cautiously, the original one, are more restricted to the amplitude of the systematics. Cosmology 3 uses parameters closest to the original and also appears to follow the behavior observed there more closely (Fig. 4.24). This technique is thus sensitive to differences in cosmic parameters, with larger parameter differences causing larger differences in such graphs tracing pixel systematics.

Practically, this result means that in this shuffling range using the Minkowski map generation parameters stated above, the amount of small structures in the unshuffled data with respect to the surrogates is cosmology dependent. All Minkowski maps show the characteristic patterns also seen in, e.g., Fig. 4.23, but the effect of the shuffling is different. In cosmology 1 and for larger-scale shuffling, the amount of small structures caught by the MT barely changes.

The cosmic parameters will also affect the thresholds used for Minkowski map creation since they are chosen in terms of the mean and standard deviation of pixel brightness (as represented by the colorbars in Fig. 4.27). I here treat this cosmology-

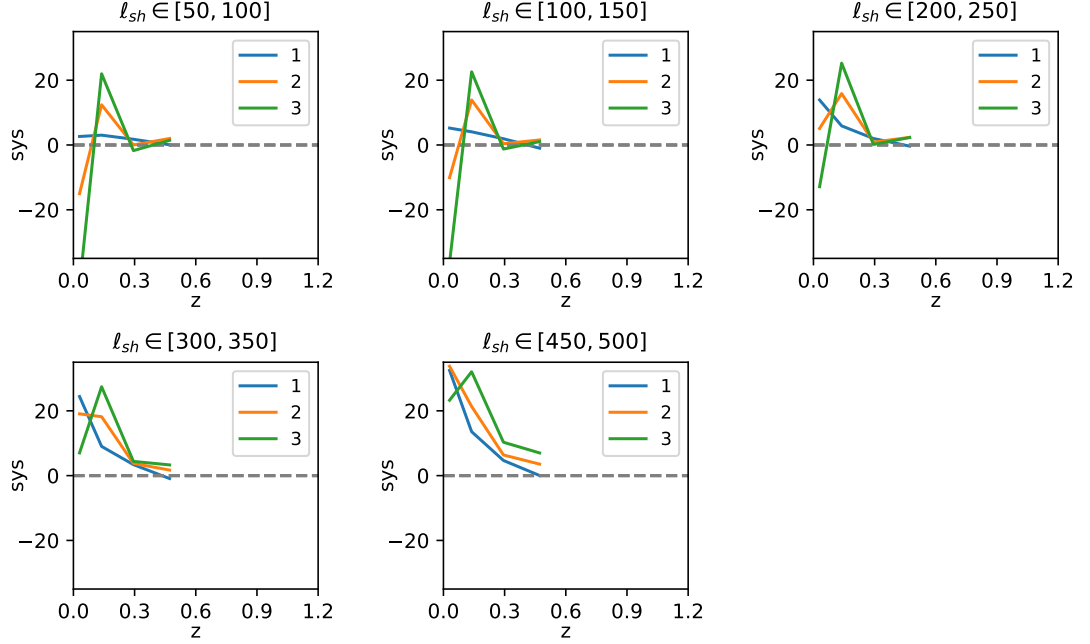


Figure 4.28: Deviation between unshuffled and surrogate EVQ maps for alternative cosmologies numbered as in Tab. 4.2 as measured by the amplitude of systematics in the deviation maps using local percentiles. Minkowski window size 1 deg for all z .

dependent variation in thresholds not as an unwanted side effect but as part of the analysis. Looking at Fig. 4.27, the low-redshift structures for cosmology 1 are, for this threshold choice, often not collapsed into round shapes following the cosmic web, but keep a more intricate structure even on smaller scales — it also has the lowest $\Omega_{m,0}$ and σ_8 of the simulations analyzed here, so slower structure formation is expected. This is traced excellently by the rank 2 MT. One hypothesis to take from here is that lower matter densities and power spectrum amplitudes lead to smaller differences in systematics between originals and surrogates at low redshift as seen here. This could be checked using more cosmologies with a finer variation in parameters.

A limiting factor of this analysis utilizing the systematics as tracers of changes in structure is their dependence on the input resolution. The redshift after which more complicated structures can be resolved into almost spherical clumps is dictated by this parameter. However, if one uses the same input N_{side} for maps to be compared, the effect will either cancel out or trace structural differences between the maps. Only regarding a single cosmology the information content is limited. Also, the limited number of redshift shells available limits any statement about peak positions, which may lie in between the redshift values presented here.

This analysis can furthermore be extended by, e.g., smoothing the maps with a kernel of either constant angular size or proportional to the comoving window size, or using higher orders. In this way, the systematics stemming from pixel shapes might be avoided, opening up different analysis paths. This work is ongoing; the state presented here reflects the state at the time of writing.

Chapter 5

Summary and Conclusions

The MT are versatile and comprehensive shape indicators that are well established in Euclidean space. There exists a Cartesian representation, calculated as an integral over either the volume or the perimeter of the shape, of a symmetric tensor product of position and normal vectors that quickly gets unwieldy to use at higher ranks. However, for the translation-invariant MT $W_1^{0,s}$ there is an irreducible representation of arbitrary rank that boils down to a single complex number containing information on amplitude and direction of any given n -fold symmetry.

In this work, I have shown a generalization of the irreducible MT to the sphere where normal vectors are interpreted as tangent space vectors with the corresponding mathematical tools. This is relevant, e.g., for the analysis of astronomical and cosmological data spread over large parts of the sky. I implemented them along with the Cartesian MT for use on HEALPix images in a software package called `litchi` that I have made publicly available. With this toolset Minkowski maps can be created that depict the morphology of local regions. Notably, masking can be done without any additional overhead by simply marking Minkowski map pixels as masked when more than a given fraction of the corresponding input pixels are masked. Alternatively, one can vary the pixel brightness threshold used to calculate the MT in a single region for threshold-dependent information.

I have shown how `litchi` can be applied to Planck CMB data analysis in chapter 3, comparing temperature maps of both data and simulations, and finding two noteworthy spots near $(l_1, b_1) = (225, -69)$ and $(l_2, b_2) = (80, 27)$ where the data is more elongated than the simulations. The calculation of Minkowski maps can be done directly on the possibly downgraded, smoothed, and masked temperature maps.

Such an analysis can be applied to any data set given in or transformed into HEALPix format. This includes LSS catalogs projected onto the sphere. I have shown simple analyses for several such data sets in chapter 4, where point sources are added to HEALPix maps. These maps are then smoothed with a radius according to the number of sources present to create an intricate pattern ready for analysis with `litchi`. Additionally, it is possible to analyze diffuse emission data like in Euclidean space (sec. 4.2.3), with the caveat that noise creates more issues due to systematics. These systematics stem from the irregular pixel shape in HEALPix and are visible primarily in the rank 2 tensors when analyzing mostly pixel-scale structures. Another caveat for data sets with fewer sources spaced further apart is the potentially higher variation in

contour length per Minkowski map pixel leading to differences between $|\psi_2|$ and the EVQ.

The aforementioned systematics can also be used as a means of data analysis, which I have shown for Magneticum simulations of LSS for different cosmologies. One can create surrogate maps, where phases of $a_{\ell m}$ in a given range in ℓ are shuffled, thereby destroying non-Gaussian information in that range. In comparison to the originals in rank 2, the systematics cancel out to varying degrees depending on cosmic parameters.

There are many further avenues to take from this work on. First and foremost, `litchi` is freely available to the scientific community for application to any spherical data. This does not need to remain restricted to cosmology or astronomy as other fields such as earth science also use data on the sphere. Trivially, any dataset not publicly available may be analyzed by the corresponding groups using `litchi`. Additionally, it may be straightforwardly possible to extend `litchi` with part of the formalism introduced by Happel et al., 2025, in particular their accumulated angle defect correction.

Furthermore, the parameters chosen for the analyses presented here can be varied substantially to focus on different morphological scales and aspects. One idea may be to consider higher ranks, the preferred directions of the corresponding anisotropies, or different window sizes. One main concern with any cosmological probe is the statistical nature of the analysis, structures in our observable universe being generated from stochastic processes and thus introducing a fundamental uncertainty into any cosmological endeavor. A way to circumvent this issue would be the collection of data from more universes. This is, however, beyond the scope of this work.

Acronyms

ACT Atacama Cosmology Telescope. 38, 42

BAO Baryonic Acoustic Oscillations. 32, 42

CDM cold dark matter. 26, 28, 32, 35, 39, 41, 42, 43, 62

CMB Cosmic Microwave Background. 1, 5, 9, 11, 12, 13, 33, 35, 36, 37, 38, 39, 41, 42, 43, 46, 47, 50, 54, 61, 62, 63, 64, 65, 70, 71, 79, 85, 103

DESI Dark Energy Spectroscopic Instrument. 32, 43

EVQ eigenvalue quotient. 49, 56, 57, 63, 65, 67, 68, 69, 79, 82, 93, 94, 95, 96, 99, 100, 101, 104, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131

FLRW Friedmann-Lemaître-Robertson-Walker. 3, 6, 18

FWHM full width half max. 12, 63, 64, 72, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 85, 89, 90

GLAM GaLAxy Mocks. 72, 73, 75, 77, 78

HEALPix Hierarchical Equal Area isoLatitude Pixelation. 54, 55, 56, 61, 62, 71, 72, 103, 112

LSS Large-Scale Structure. 1, 19, 23, 33, 41, 43, 71, 72, 73, 103, 104

MF Minkowski functionals. 1, 42, 45, 46, 47, 61, 62, 71

MT Minkowski tensors. 1, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 58, 59, 61, 62, 70, 71, 73, 85, 87, 91, 100, 101, 103, 109, 112

SDSS Sloan Digital Sky Survey. 72, 76

SZ Sunyaev-Zeldovich. 37, 79

WMAP Wilkinson Microwave Anisotropy Probe. 38, 39, 41, 61, 91, 96, 98, 99

Appendix A

Rank 4 anisotropy measure examples

The figures in this appendix chapter stem from the supplementary material in Collischon et al., 2024 (Supp. Figs. 5 to 8) and show various ways to obtain scalar values from the Cartesian rank 4 MT $W_1^{0,4}$ as expressed in terms of the matrix 2.15. The middle eigenvalue, which was eventually chosen as the best anisotropy measure, is shown in Fig. 2.8.

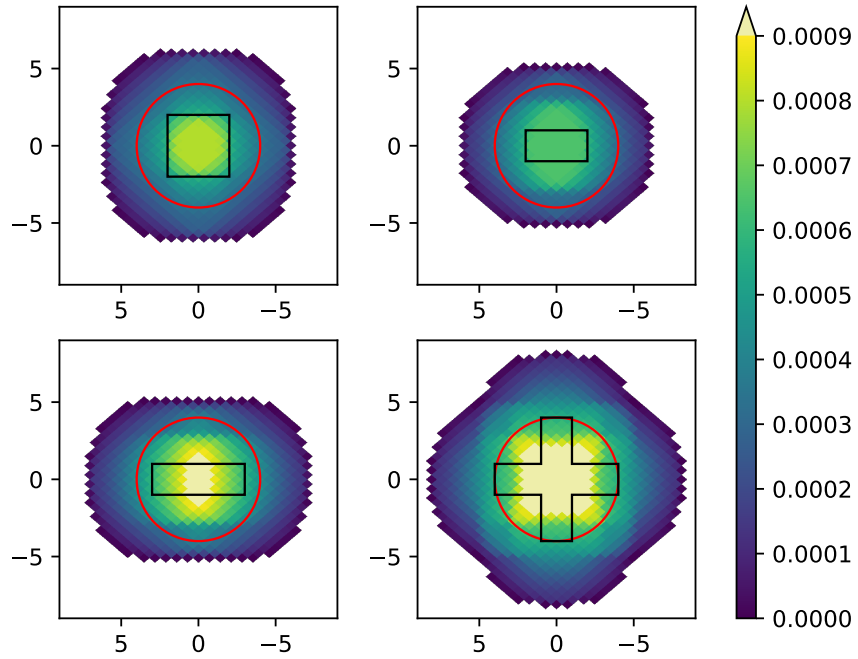


Figure A.1: Collischon et al., 2024, Supp. Fig. 5: “Minkowski maps of the length of the vector of eigenvalues of $W_1^{0,4}$ of a square, elongated rectangle, longer rectangle, and cross. The black shapes are the shape of the input body, the red circles show the window size. Coordinates are given in deg. The value of N_{side} of the input was 512, that of the output is 128.”

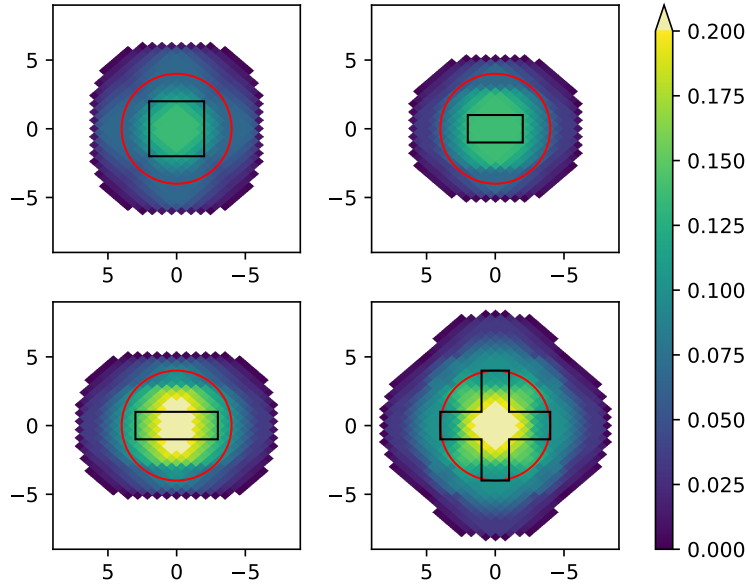


Figure A.2: Collischon et al., 2024, Supp. Fig. 6: “Minkowski maps of the largest eigenvalue of $W_1^{0,4}$ of a square, elongated rectangle, longer rectangle, and cross. The black shapes are the shape of the input body, the red circles show the window size. Coordinates are given in deg. The value of N_{side} of the input was 512, that of the output is 128.”

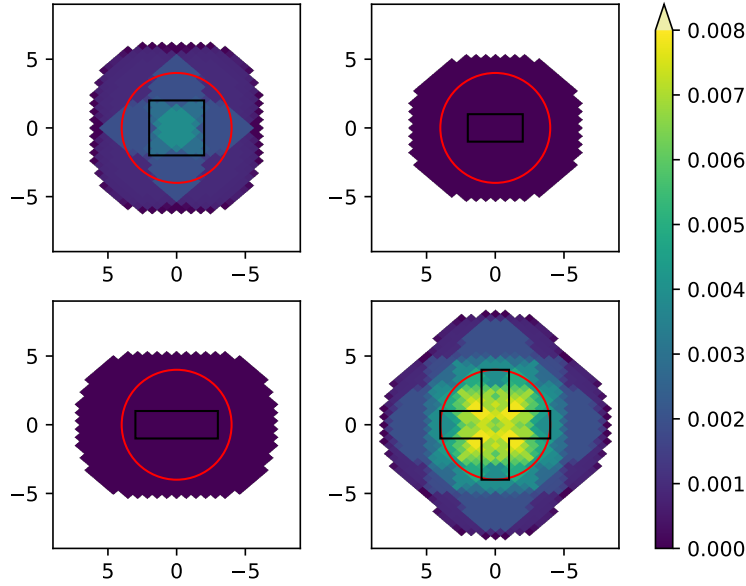


Figure A.3: Collischon et al., 2024, Supp. Fig. 7: “Minkowski maps of the smallest eigenvalue of $W_1^{0,4}$ of a square, elongated rectangle, longer rectangle, and cross. The black shapes are the shape of the input body, the red circles show the window size. Coordinates are given in deg. The value of N_{side} of the input was 512, that of the output is 128.”

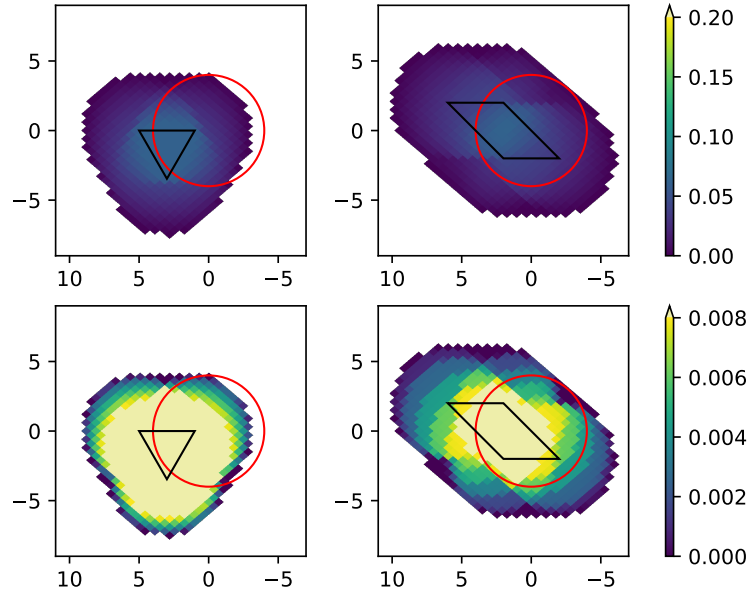


Figure A.4: Collischon et al., 2024, Supp. Fig. 8: “Minkowski maps of the middle (top) and smallest (bottom) eigenvalue of $W_1^{0,4}$ of a triangle and parallelogram. The black shapes are the shape of the input body, the red circles show the window size. Coordinates are given in deg. The value of N_{side} of the input was 512, that of the output is 128. Colorbars are the same as in the respective images above.”

By reading this thesis, you do not agree to the collection of personal data. No cookies are used. If the reader would like some cookies, they can be produced in various ways. One proven example works as follows: Thoroughly mix 125 g of soft butter and 125 g of sugar. The butter may be softened using artificial microwave radiation (the CMB is not sufficient) and caution due to spontaneous explosive events is advised here. One egg is added and a homogeneous substance produced. Then, a mix of 140 g of flour type 405 or 550, half a teaspoon of each salt and baking soda, and 100 g of chocolate chips is added carefully, as mixing too much will affect the final product’s structural properties. The type of flour will also affect structural integrity, but both types mentioned here will lead to satisfying results. Place walnut-sized portions on baking sheet and bake at 190 deg C until slight browning of the edges is observed. The cookies may be deleted without affecting the functionality of this thesis.

Further examples

The capabilities of the MT in determining orientation are further demonstrated in the exemplary depiction of a rotating fish (Fig. A.5). I have created irreducible rank 2 Minkowski maps of the image of both amplitude and direction for a window size of about 10 deg, which covers significant fractions of the shape. It can be seen that while the amplitude stays the same up to rotation and pixel effects, the phase changes by an almost constant offset.

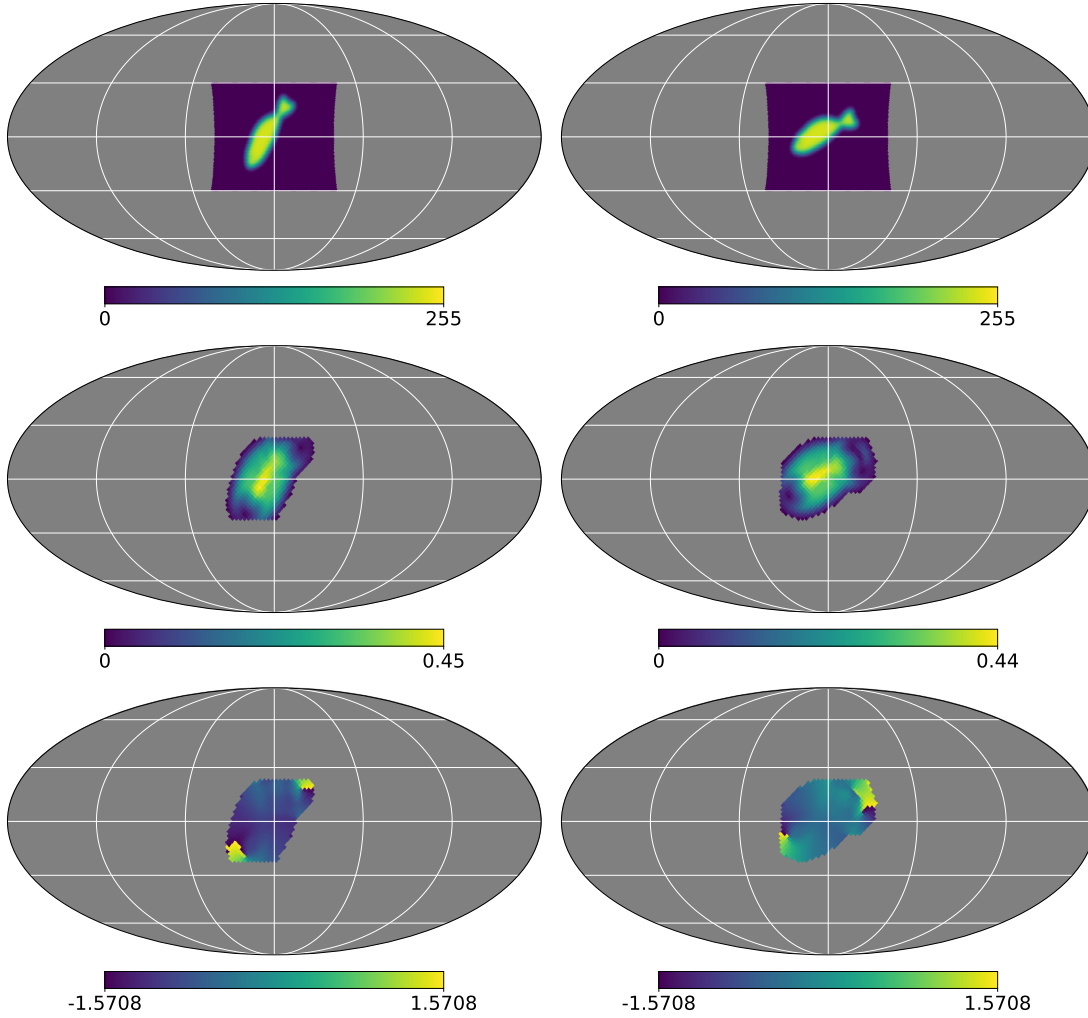


Figure A.5: Top: HEALPix format images of a fish in two orientations. Center: Irreducible rank 2 anisotropy of the images above. Bottom: Irreducible rank 2 direction of the images.

Appendix B

Magneticum surrogates

B.1 Window size 1 deg

The following figures show deviation between Magneticum simulation data at different redshift and phase-shuffled surrogates for different intervals in l (see sec. 4.2.5) and a window size of 1 deg. “sys” stands for systematics (comp. Fig. 2.6) as measured by difference between the mean pixel value up to 50 deg from the poles and the mean pixel value up to 25 deg from the equator.

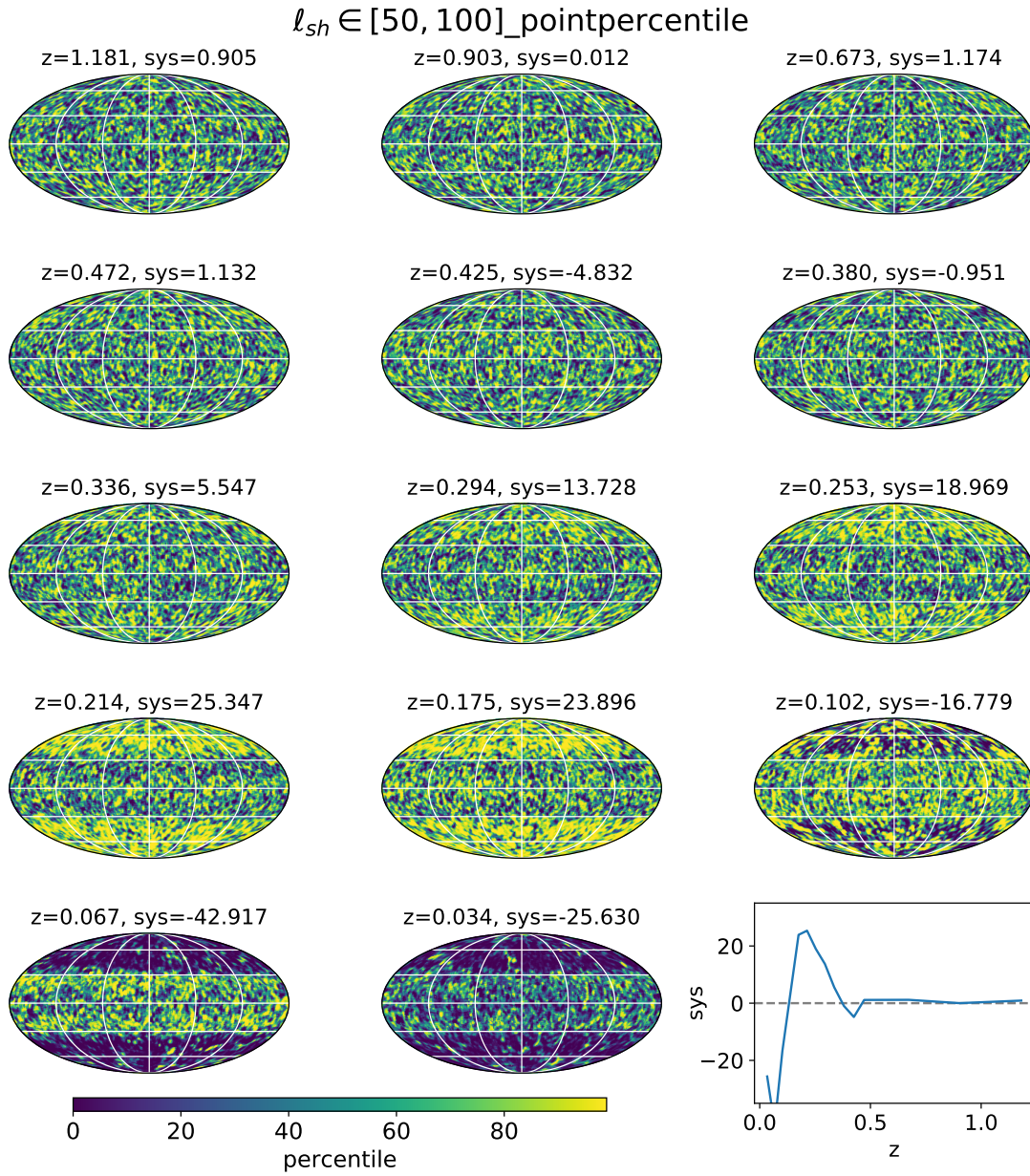


Figure B.1: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [50, 100]$. Minkowski window size 1 deg for all z .

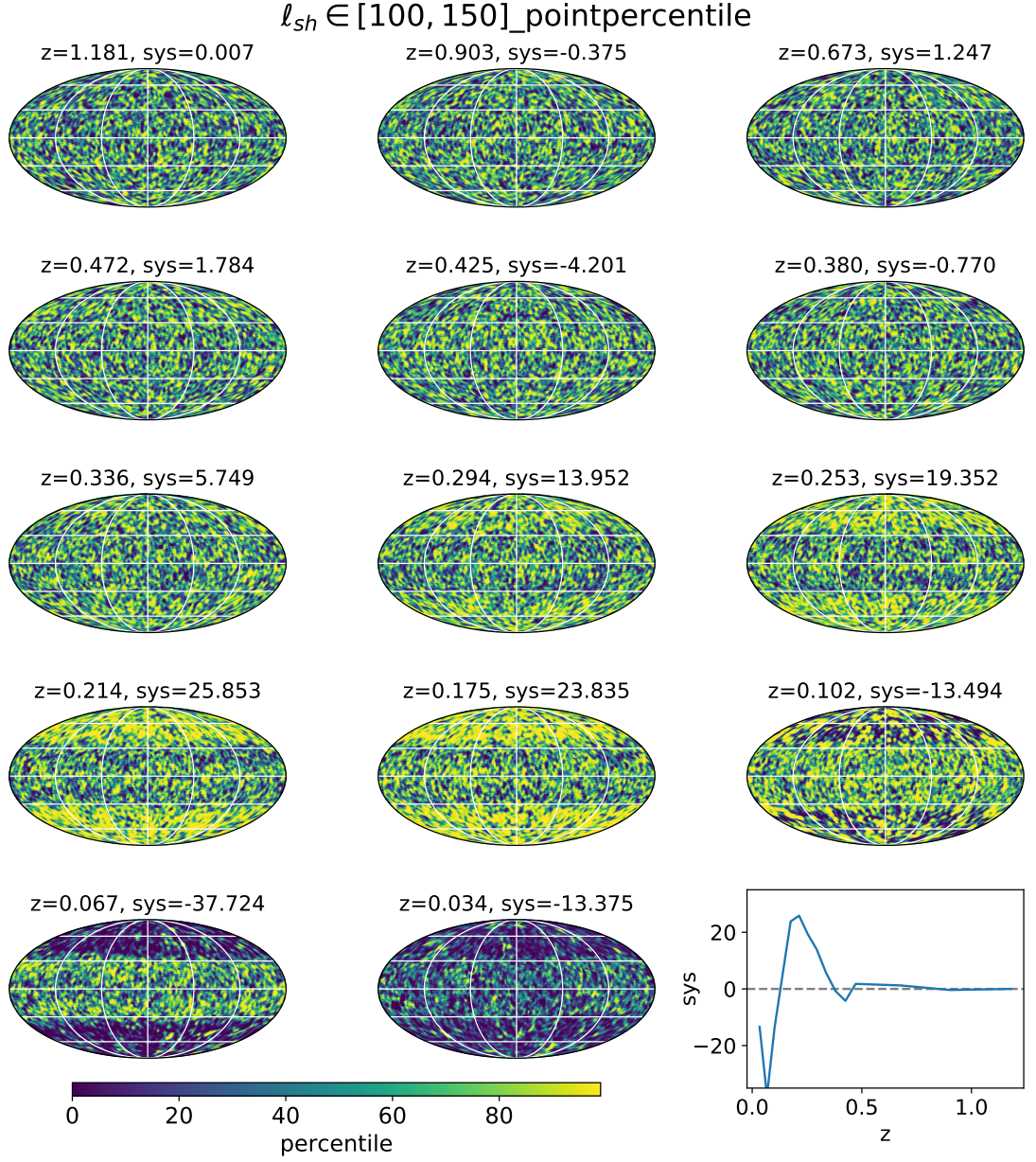


Figure B.2: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [100, 150]$. Minkowski window size 1 deg for all z .

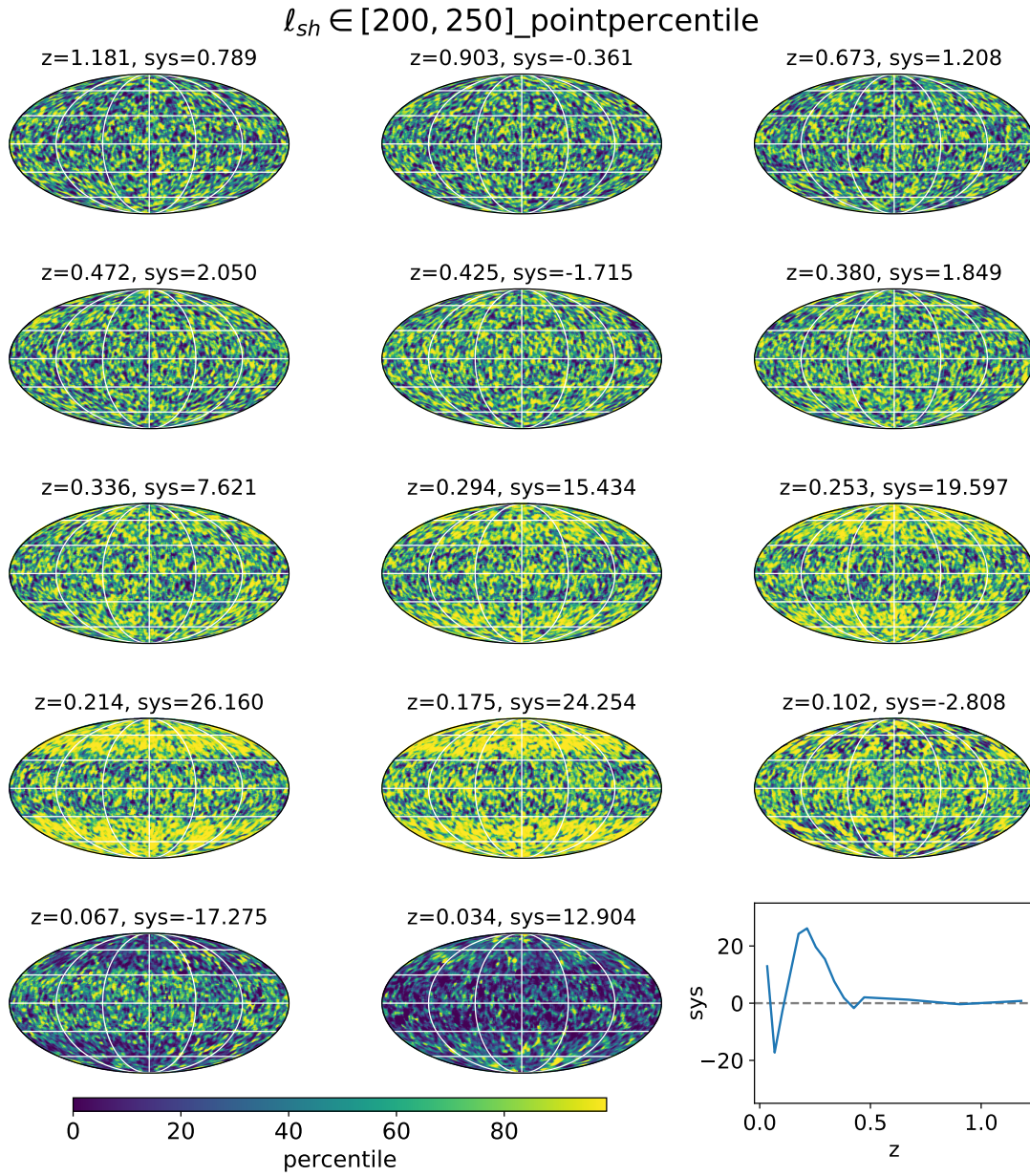


Figure B.3: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [200, 250]$. Minkowski window size 1 deg for all z .

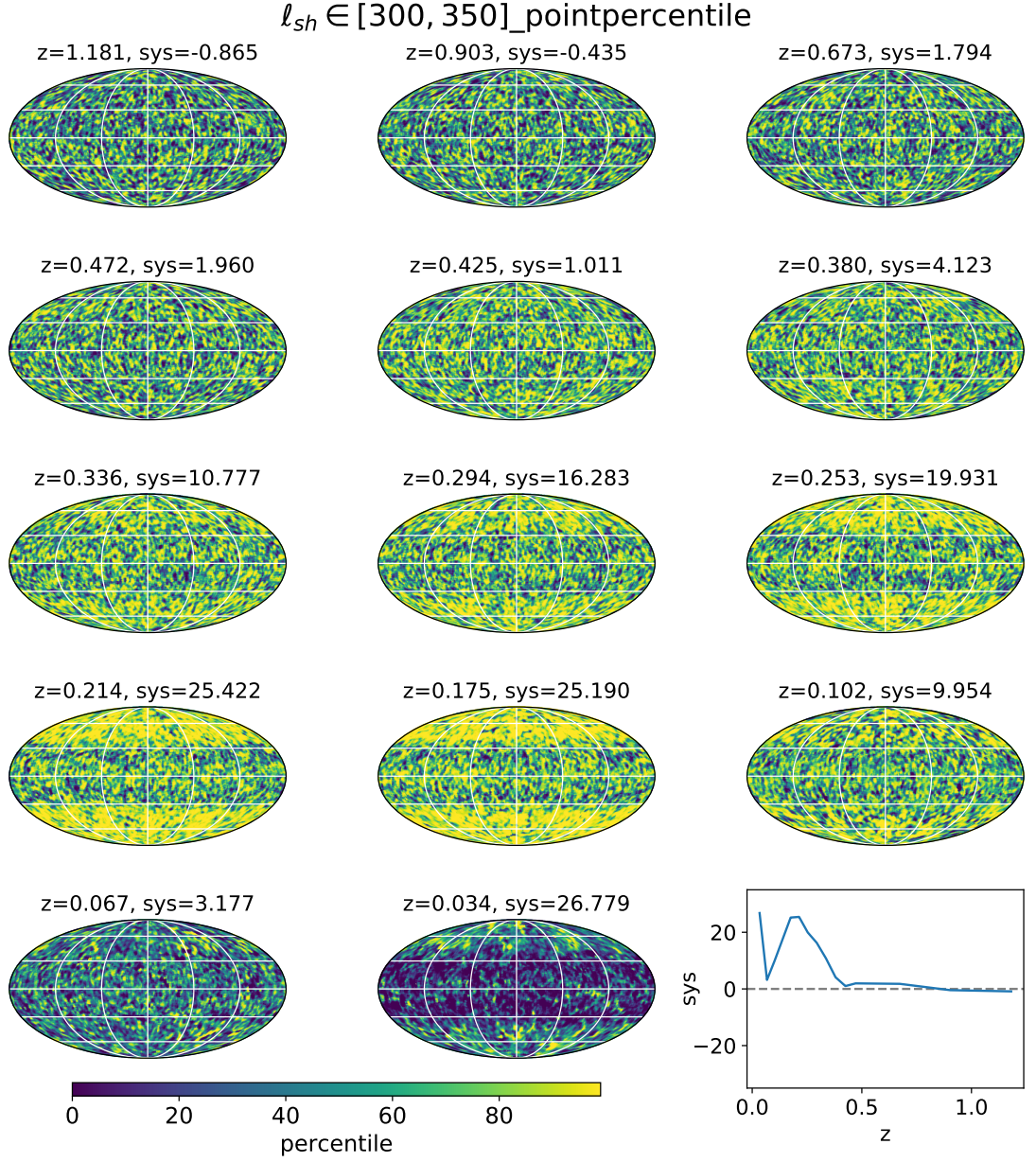


Figure B.4: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [300, 350]$. Minkowski window size 1 deg for all z .

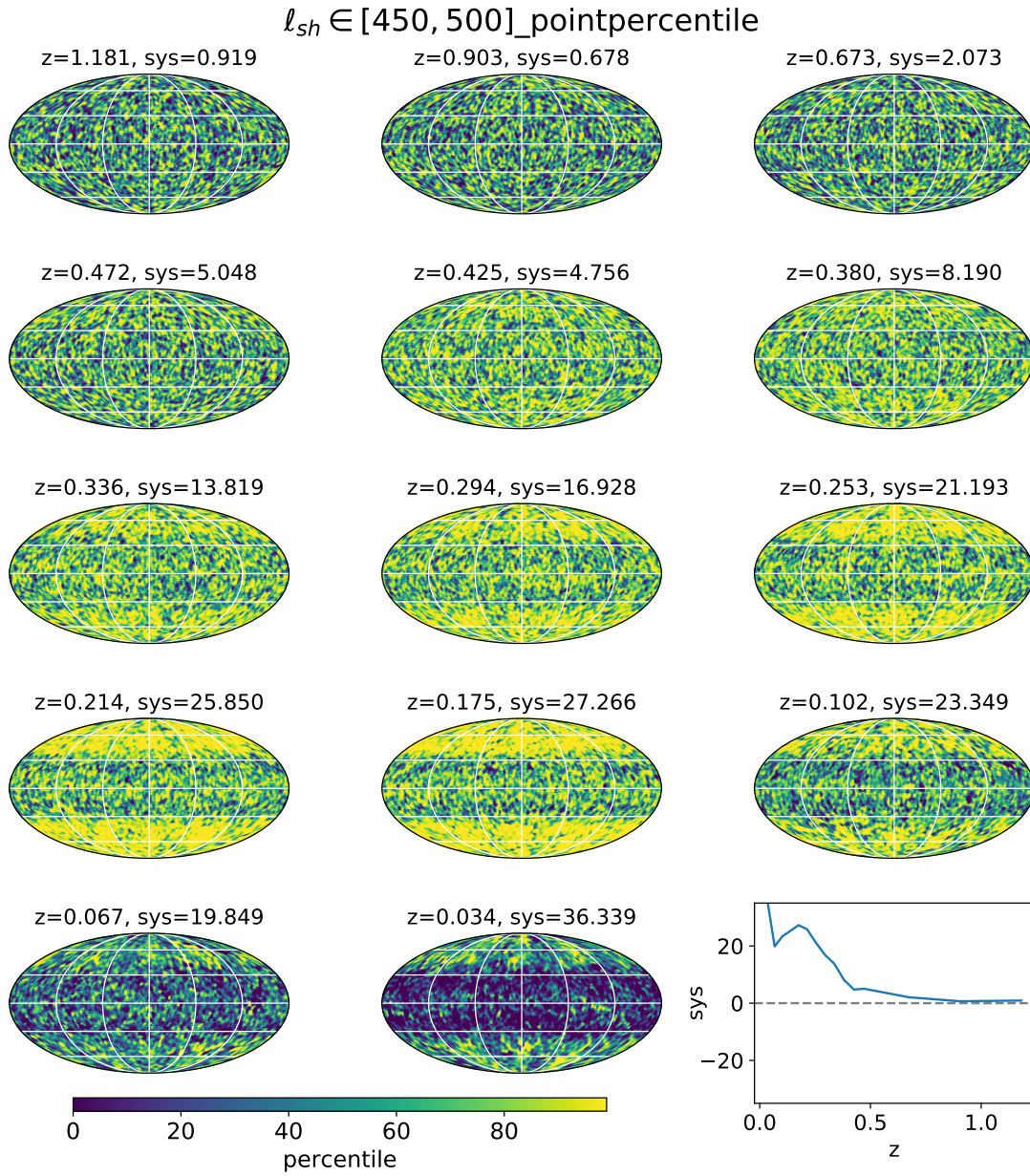


Figure B.5: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [450, 500]$. Minkowski window size 1 deg for all z .

B.2 Window size 30 Mpc

The following figures show deviation between Magneticum simulation data at different redshift and phase-shuffled surrogates for different intervals in l (see sec. 4.2.5) and a window size corresponding to 30 Mpc comoving distance. “sys” stands for systematics (comp. Fig. 2.6) as measured by difference between the mean pixel value up to 50 deg from the poles and the mean pixel value up to 25 deg from the equator.

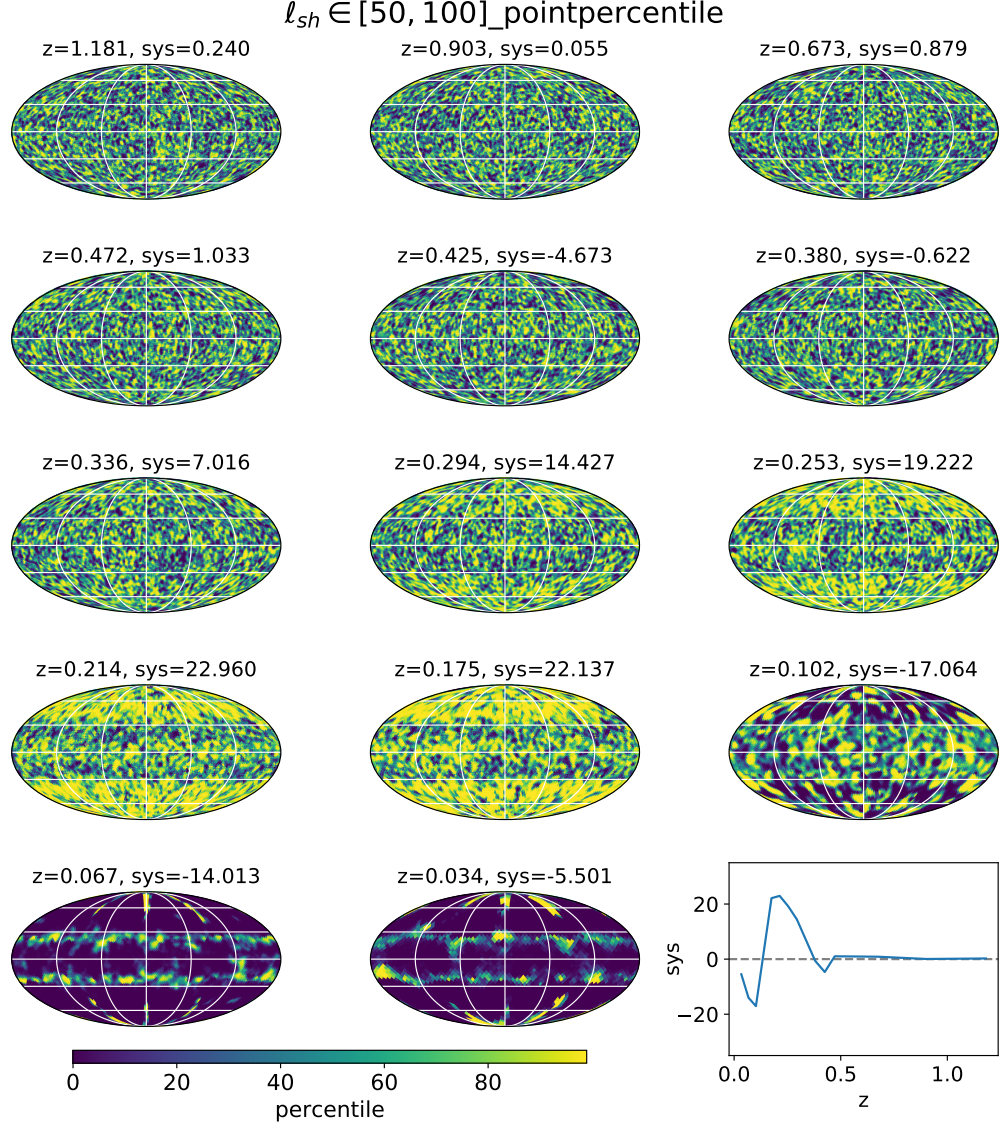


Figure B.6: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [50, 100]$. Minkowski window radius 30 Mpc comoving for all z .

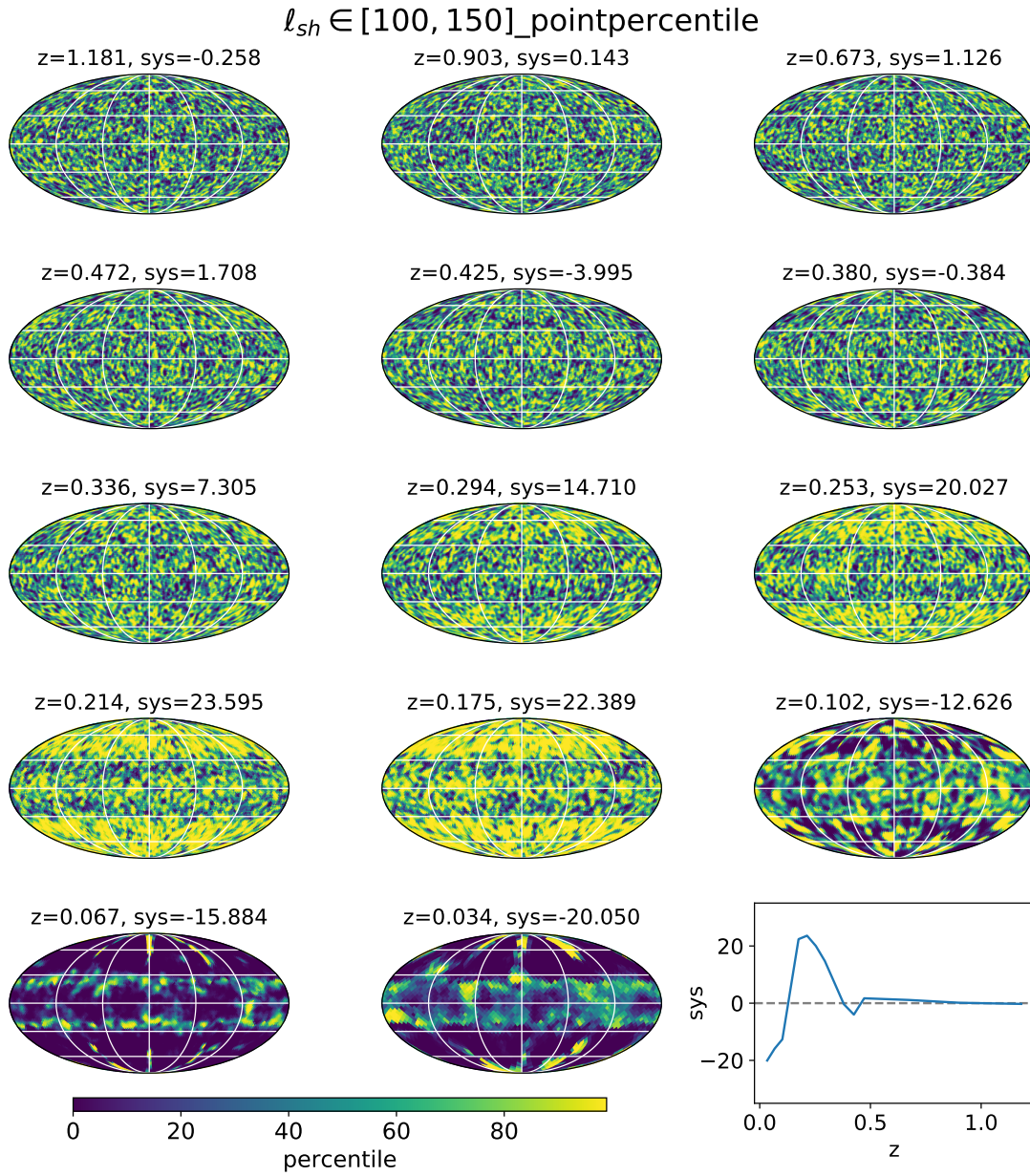


Figure B.7: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [100, 150]$. Minkowski window radius 30 Mpc comoving for all z .

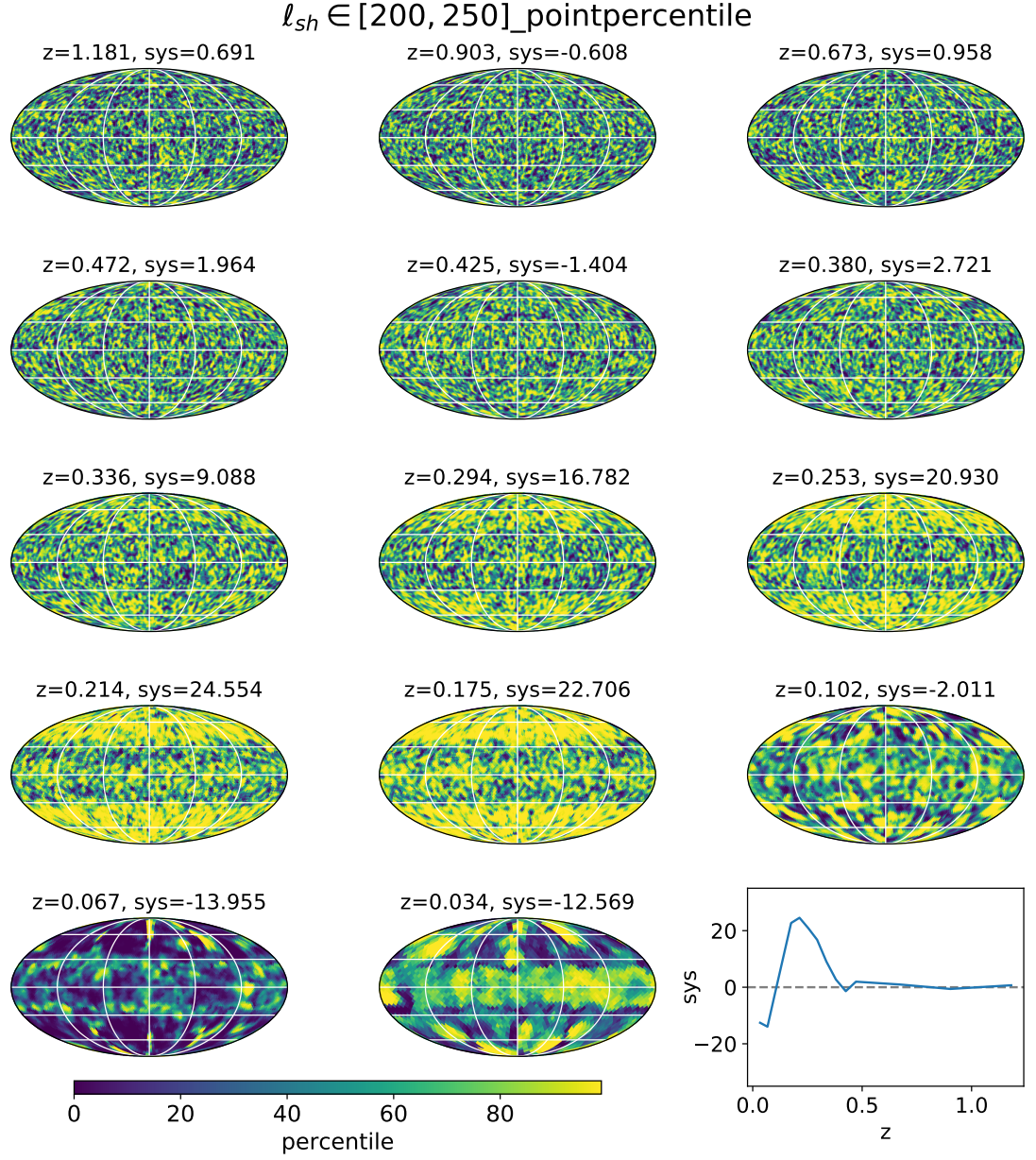


Figure B.8: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [200, 250]$. Minkowski window radius 30 Mpc comoving for all z .

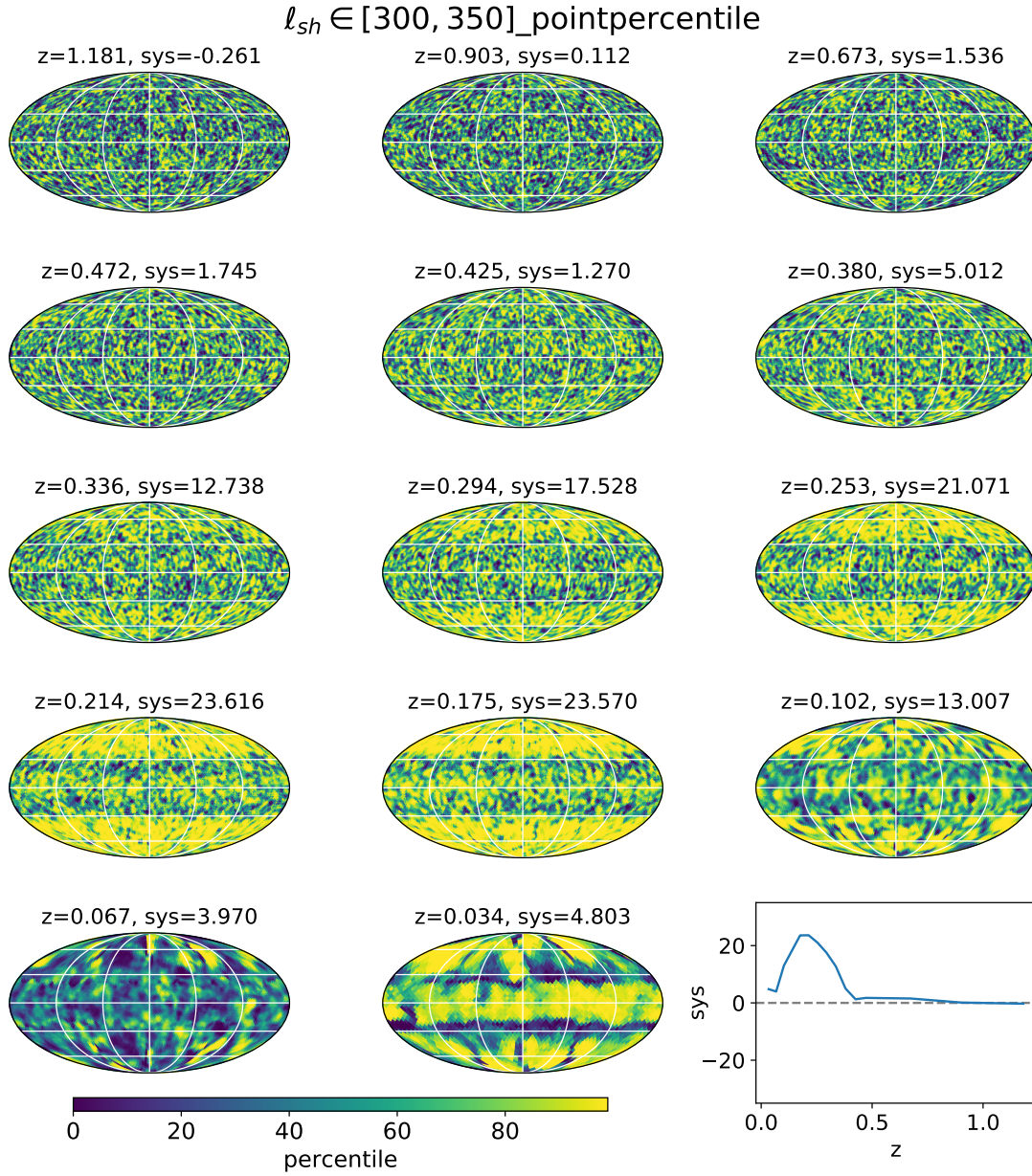


Figure B.9: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [300, 350]$. Minkowski window radius 30 Mpc comoving for all z .

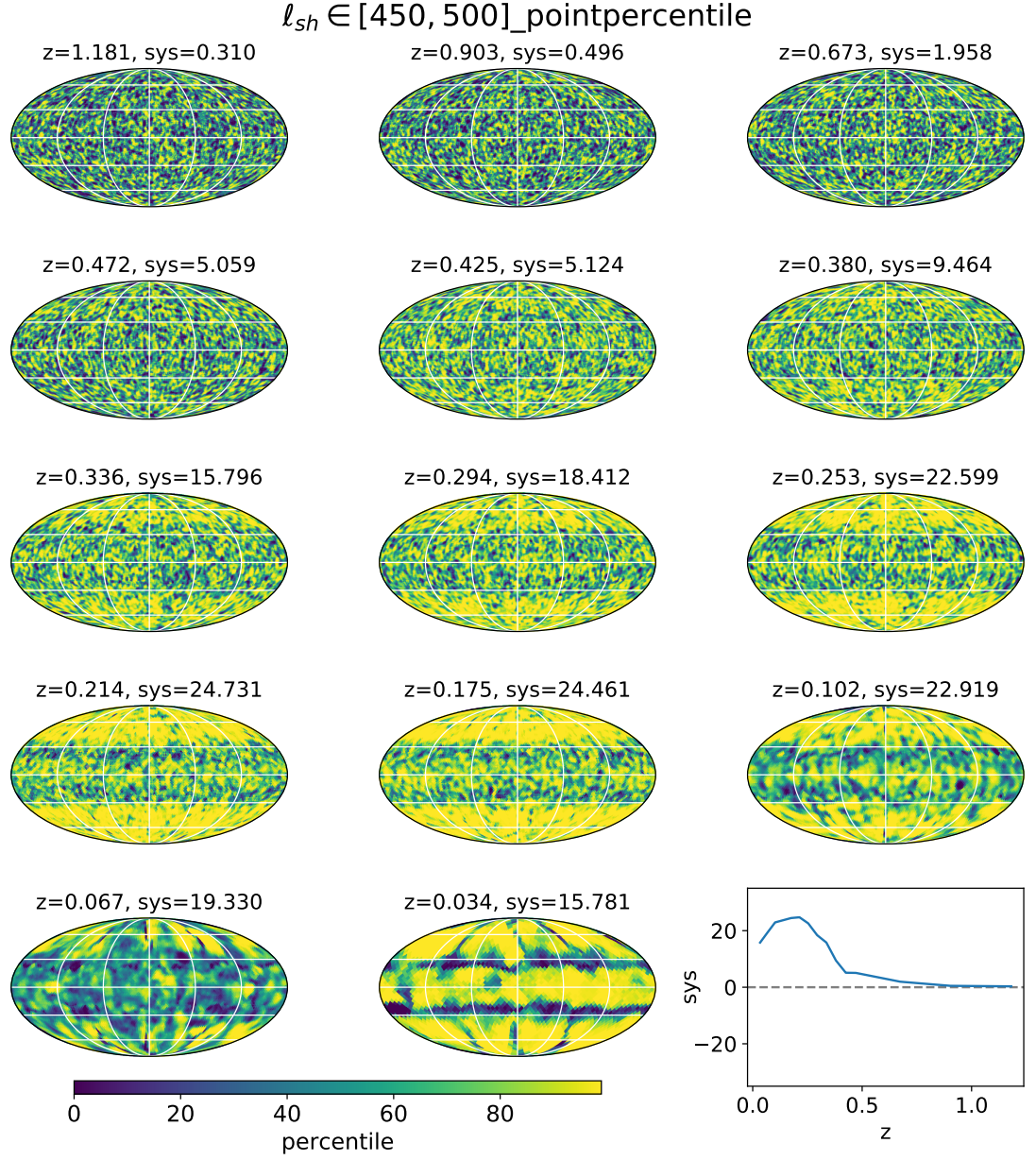


Figure B.10: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [450, 500]$. Minkowski window radius 30 Mpc comoving for all z .

B.3 Other cosmologies

The following figures show deviation between Magneticum simulation data generated for different cosmological parameters as shown in Tab. 4.2 at different redshift and phase-shuffled surrogates for different intervals in l (see sec. 4.2.5) and a window size corresponding to 30 Mpc comoving distance. “sys” stands for systematics (comp. Fig. 2.6) as measured by difference between the mean pixel value up to 50 deg from the poles and the mean pixel value up to 25 deg from the equator.

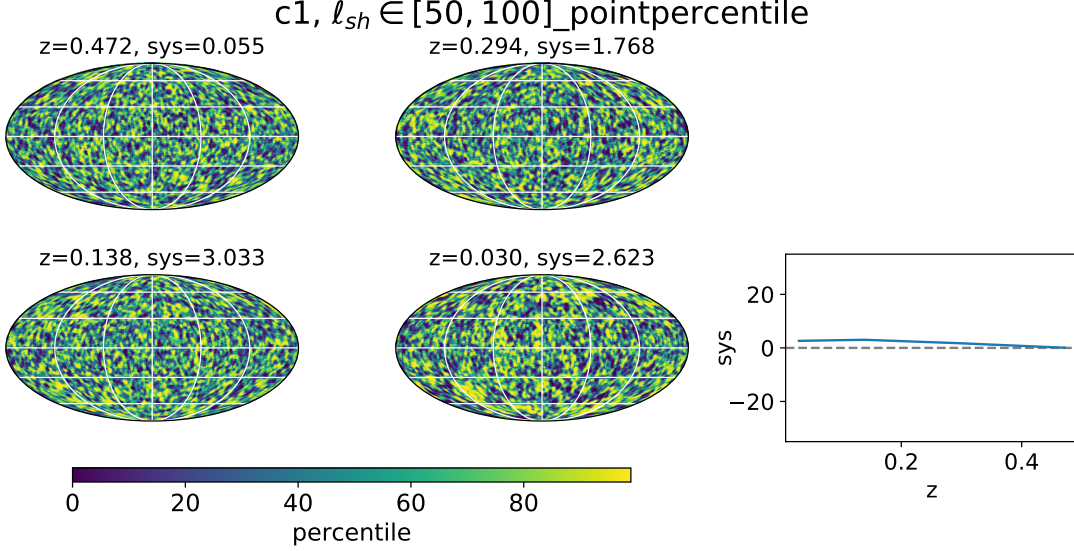


Figure B.11: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 1 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [50, 100]$. Minkowski window size 1 deg for all z .

Image analysis requires nourishment on the executing analyst’s side. This can be achieved using commonly known techniques and methods, where acquiring new methods may enhance the results. One such method that proved successful shall now be presented here. Cut a bundle of spring onions and 5 cloves of garlic. 5 teaspoons of yellow Thai curry paste are heated in oil (e.g. sesame seed oil may add notes to the flavor profile). This is done in a stainless steel container, making sure the spices and oil are not overheated. Care needs to be taken in selecting the curry paste within the analyst’s pain tolerance limits. The white part of the spring onions is added to the mix, followed by the minced garlic; the exact amounts may be adapted according to individual experience. When the onion is softened, add 100 g of cashews, followed by a small can (140 g) of chickpeas and 400 ml of coconut milk. The meal is rounded off with one can (255 g) of litchis, including a dash of the canned juice and the remainder of the spring onions. Add salt to taste. The mix is left to simmer for a few minutes and served with rice. An experimental assembly of the above was performed in collaboration with S. Tripathi. The exact amounts of each ingredient are highly dependent on local preferences with a wide region of stability, where the result is perceived as sufficient. Testing this range is, however, beyond the scope of this work. I predict that the addition of lemon juice and a small amount of zest may improve the experience in future trials.

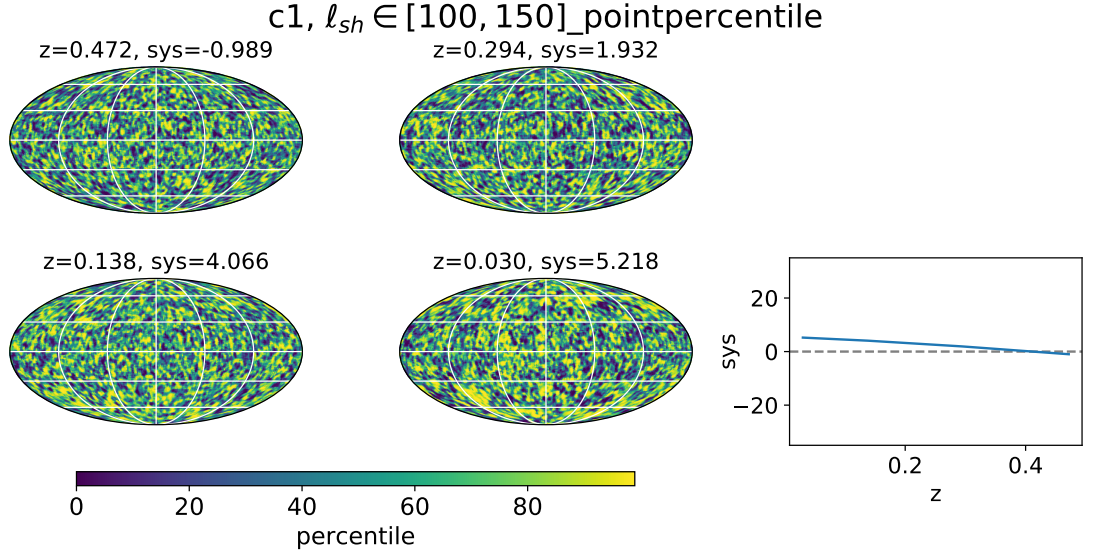


Figure B.12: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 1 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [100, 150]$. Minkowski window size 1 deg for all z .

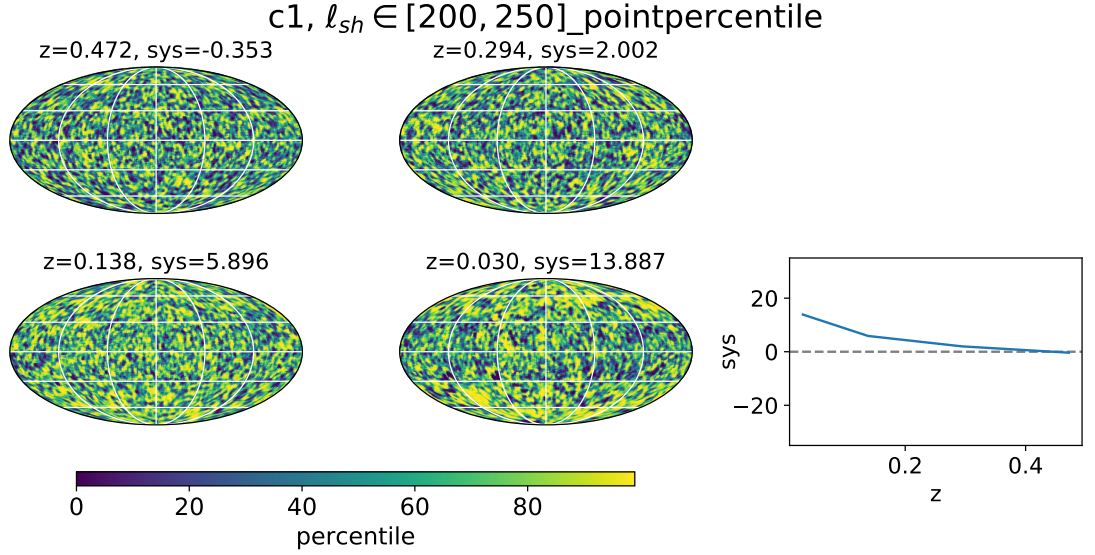


Figure B.13: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 1 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [200, 250]$. Minkowski window size 1 deg for all z .

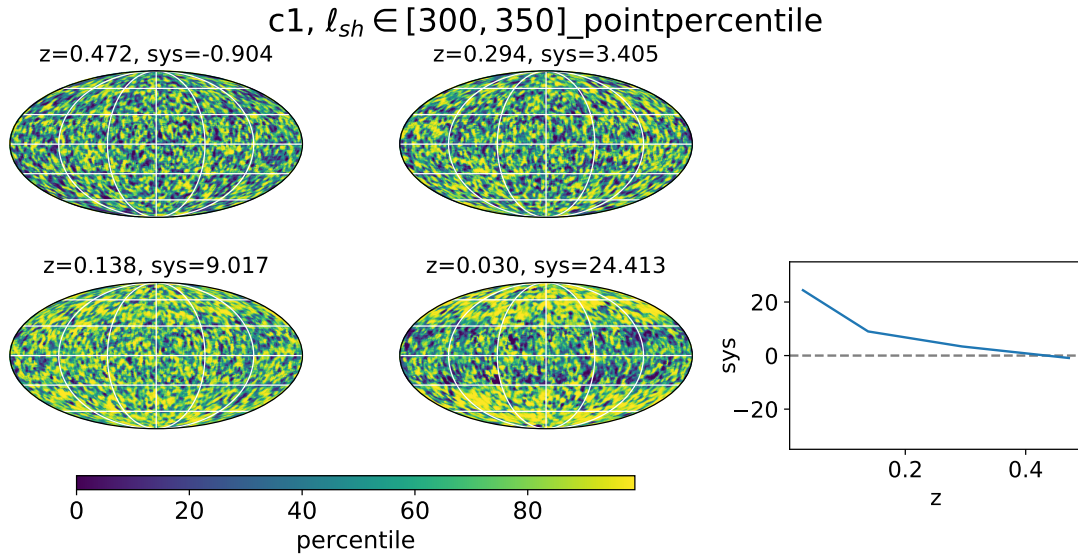


Figure B.14: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 1 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [300, 350]$. Minkowski window size 1 deg for all z .

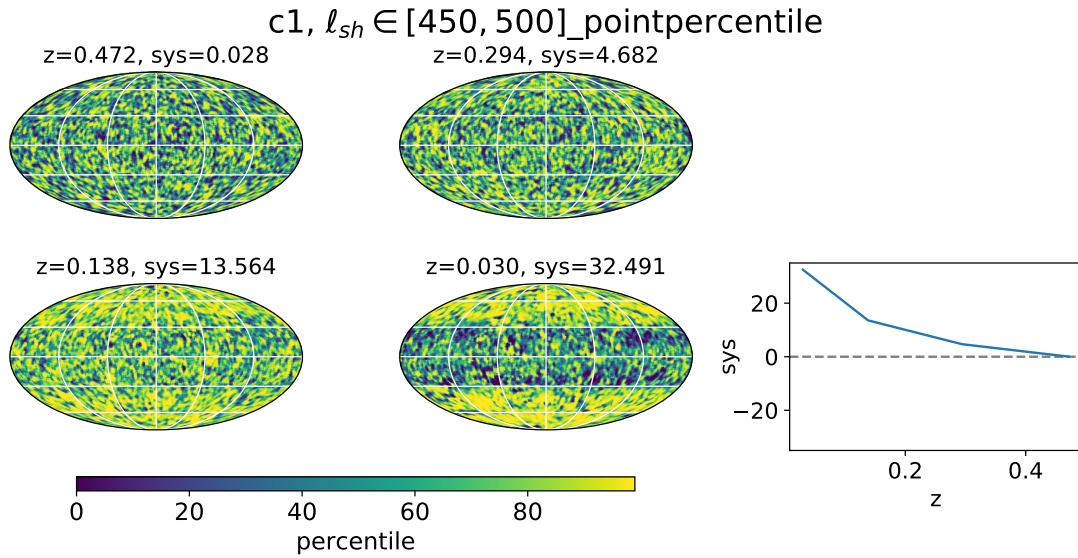


Figure B.15: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 1 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [450, 500]$. Minkowski window size 1 deg for all z .

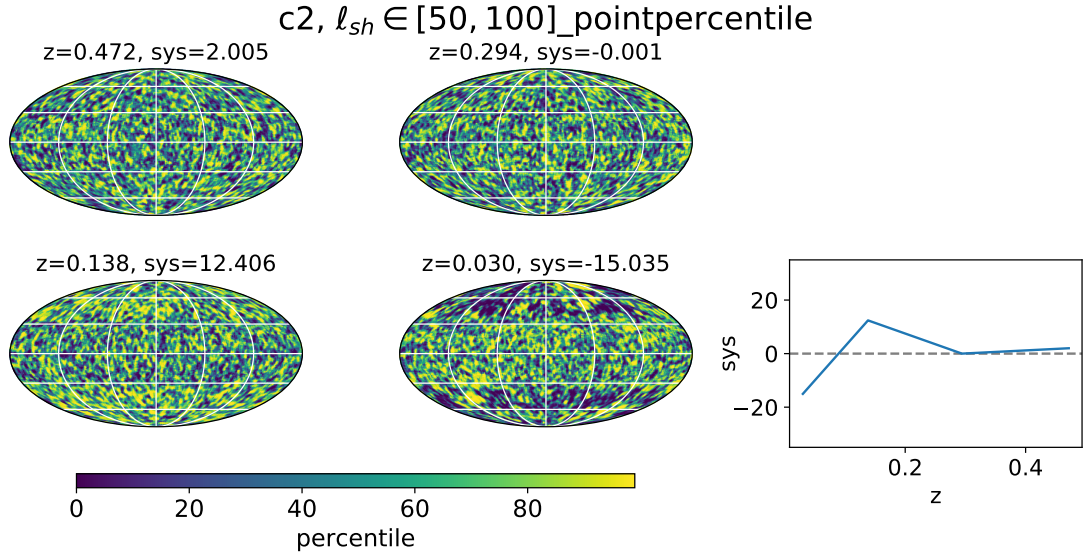


Figure B.16: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 2 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [50, 100]$. Minkowski window size 1 deg for all z .

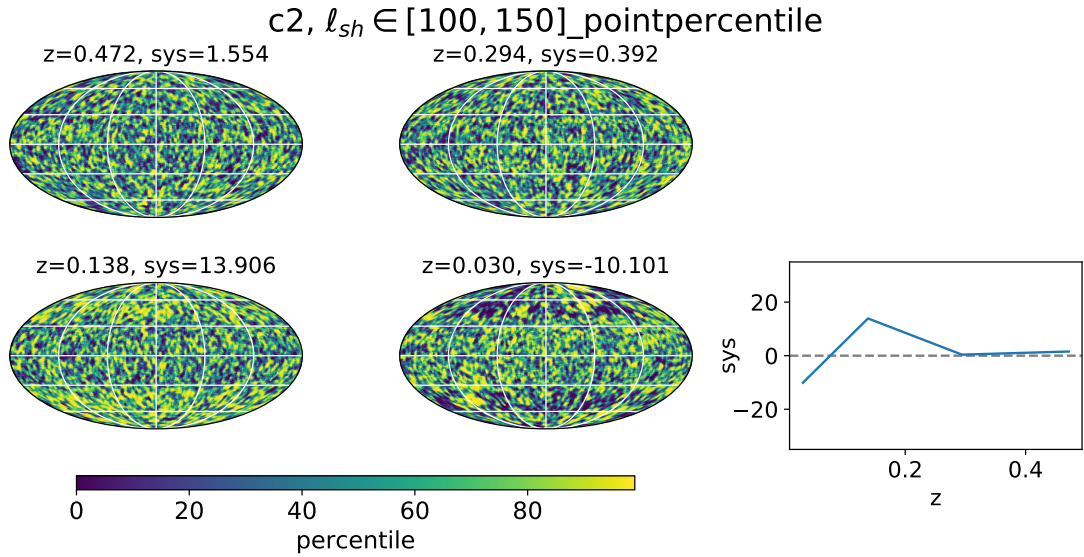


Figure B.17: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 2 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [100, 150]$. Minkowski window size 1 deg for all z .

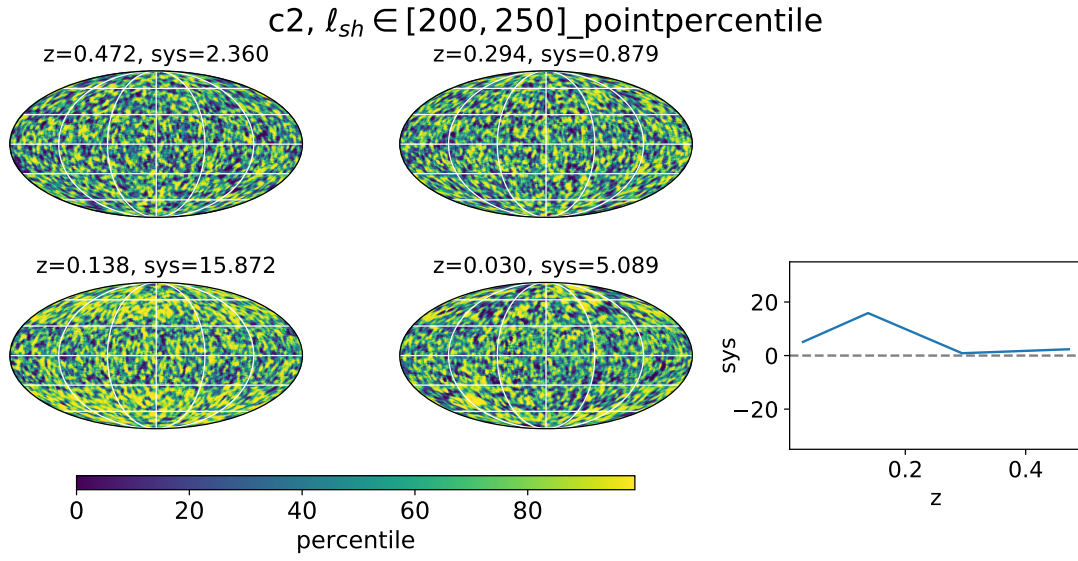


Figure B.18: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 2 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [200, 250]$. Minkowski window size 1 deg for all z .

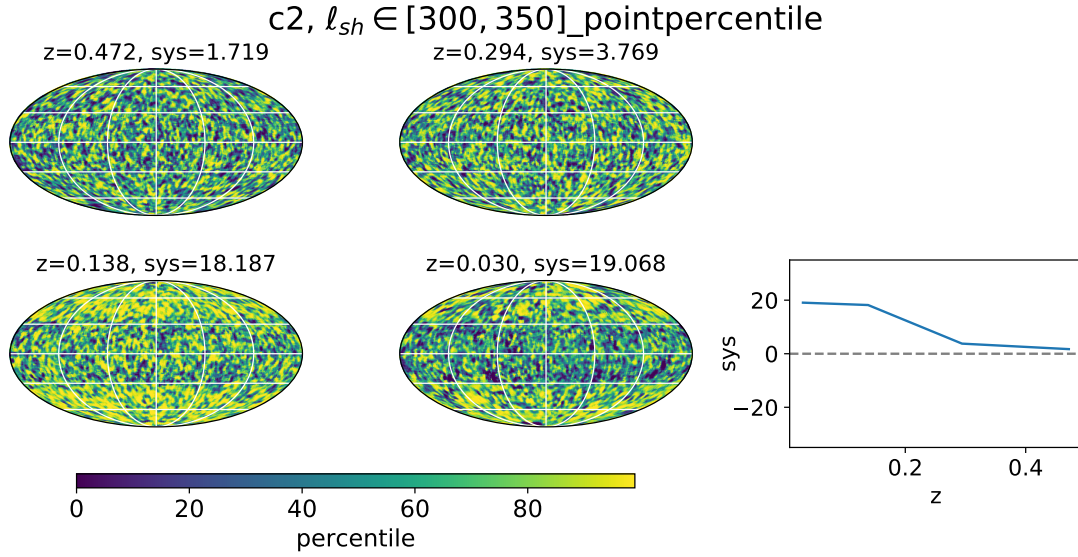


Figure B.19: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 2 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [300, 350]$. Minkowski window size 1 deg for all z .

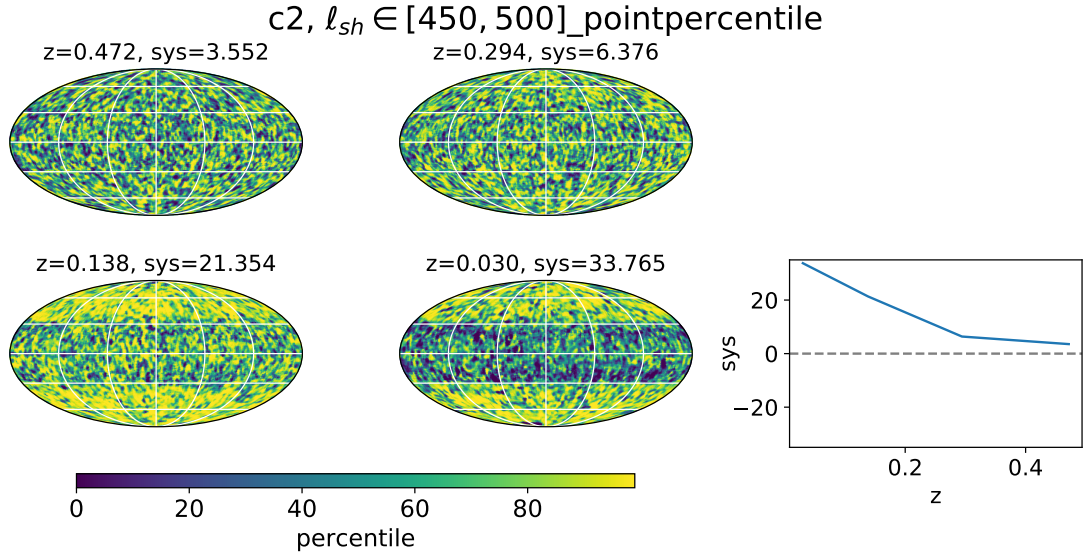


Figure B.20: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 2 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [450, 500]$. Minkowski window size 1 deg for all z .

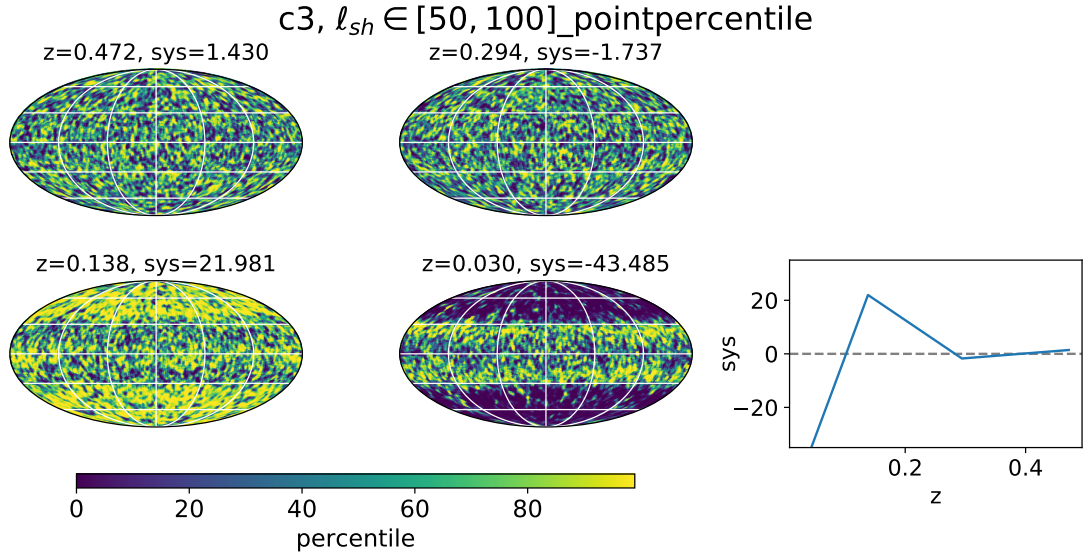


Figure B.21: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 3 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $l_{sh} \in [50, 100]$. Minkowski window size 1 deg for all z .

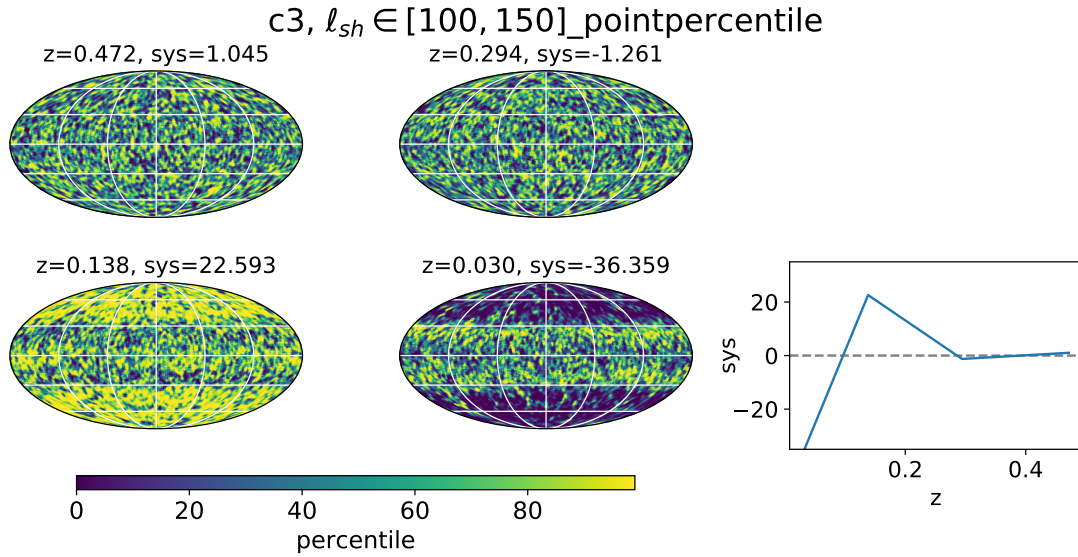


Figure B.22: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 3 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [100, 150]$. Minkowski window size 1 deg for all z .

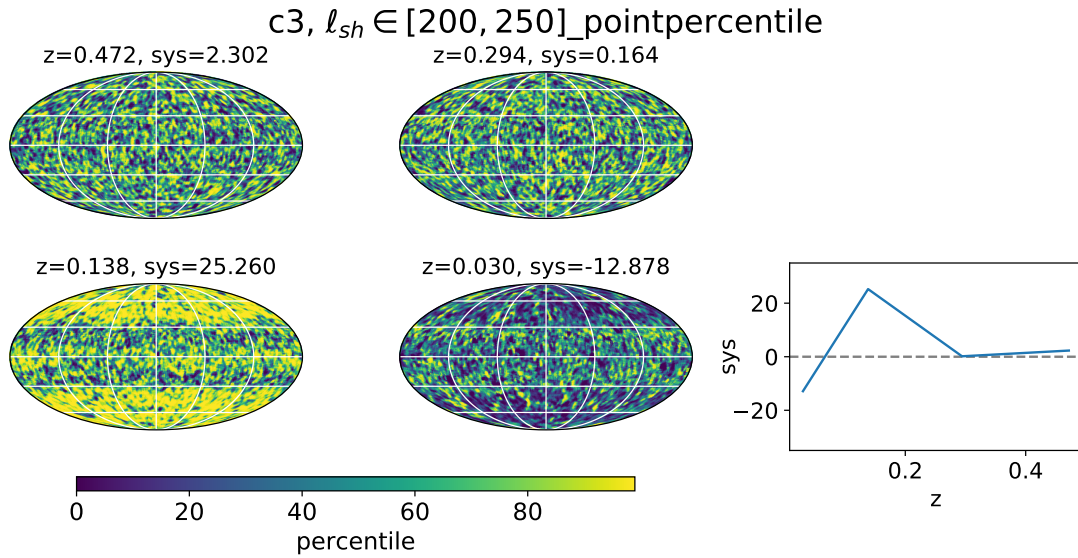


Figure B.23: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 3 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [200, 250]$. Minkowski window size 1 deg for all z .

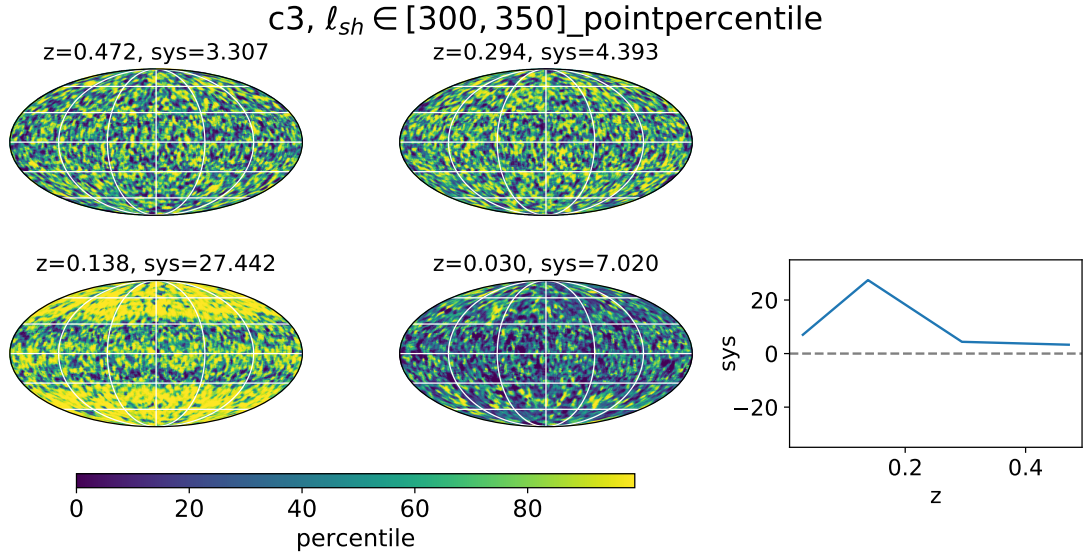


Figure B.24: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 3 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [300, 350]$. Minkowski window size 1 deg for all z .

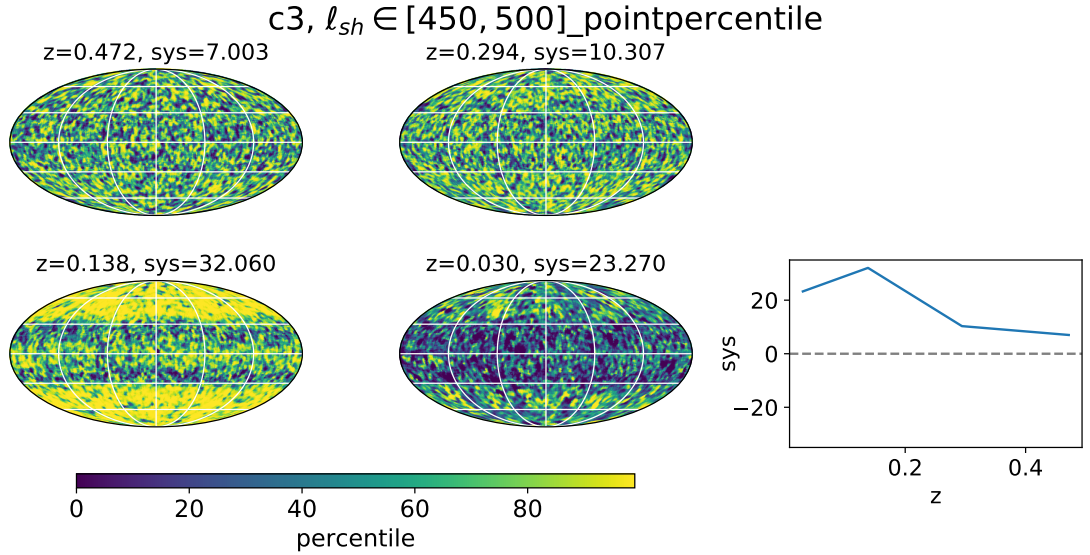


Figure B.25: Deviation between EVQ Minkowski maps of Magneticum data and surrogates (see sec. 4.2.5) using cosmology 3 (see Tab. 4.2) for varying redshift as measured by local percentile of the data maps with respect to the surrogates. Here: surrogates with phases shuffled in $\ell_{sh} \in [450, 500]$. Minkowski window size 1 deg for all z .

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