Lösung zur Analysis II - Klausur für Physiker

Aufgabe 1

a, $s(\gamma) = \int_0^1 \left|\dot{\gamma(t)}\right| dt$; $\frac{d\gamma(t)}{dt} = \begin{pmatrix} 3t^2 \\ -4t \end{pmatrix}$; $\left|\dot{\gamma(t)}\right| = \sqrt{9t^4 + 16t^2} \overset{t \geq 0}{\equiv} t\sqrt{9t^2+16}$

Sei $u = 16 + 9t^2$; $\frac{du}{dt} = 18t$; $dt = \frac{du}{18t}$; $\int_0^1 t\sqrt{9t^2+16}\, dt = \int_{16}^{25} \frac{\sqrt{u}}{18}\, du = \frac{1}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{16}^{25} = \frac{61}{27}$

b, Sei $(3t^2, -4t) = \lambda(1,-2) \quad \Rightarrow 3t^2 = \lambda$ und $-4t = \lambda(-2)$
$\Rightarrow \lambda = 2t \quad \Rightarrow 3t^2 = 2t \quad \Rightarrow t = \frac{2}{3} \quad \Rightarrow \quad (x_0, y_0) = (\frac{8}{27}, \frac{10}{9})$
c,

d, Sei $x = t^3 \Rightarrow t = \sqrt[3]{x} \quad \Rightarrow 2 - 2t^2 = 2 - 2x^{\frac{2}{3}} \quad \Rightarrow \int_0^1 2 - 2x^{\frac{2}{3}}\, dx = \left[2x - \frac{6}{5}x^{\frac{5}{3}}\right]_0^1 = \frac{4}{5}$

Aufgabe 2

a, $J := \begin{pmatrix} a\sin(\Theta)\cos(\varphi) & ar\cos(\Theta)\cos(\varphi) & -ar\sin(\Theta)\sin(\varphi) \\ b\sin(\Theta)\sin(\varphi) & br\cos(\Theta)\sin(\varphi) & br\sin(\Theta)\cos(\varphi) \\ c\cos(\Theta) & -cr\sin(\varphi) & 0 \end{pmatrix}$

$\Rightarrow \quad \det J = abcr^2\sin(\Theta)$

b, $f^{-1} : \mathbb{R}^3 \quad \rightarrow \quad \mathbb{R}^3$
$f^{-1} : (x,y,z) \quad \mapsto \quad (\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}, \arccos(\frac{z}{cr}), \arctan(\frac{bx}{ay}))$

Denn :

$$\sqrt{\frac{a^2r^2\sin(\Theta)^2\cos(\varphi)^2}{a^2} + \frac{b^2r^2\sin(\Theta)^2\sin(\varphi)^2}{b^2} + \frac{c^2r^2\cos(\Theta)^2}{c^2}} = r$$

$$\arccos(\frac{cr\cos\Theta)}{cr}) = \Theta$$

$$\arctan(\frac{abr\sin(\Theta)\sin(\varphi)}{abr\sin(\Theta)\cos(\varphi)}) = \varphi$$

Umkehrung analog zu Übungsblatt 6 Aufgabe 2!

Aufgabe 3

a, $D_f(x,y,z) = (2x+1,\, 6y^2+6y-2z-1,\, 4z-2y)$
Sei nun $D_f(x,y,z) = 0_V \quad \Rightarrow \quad x = -\frac{1}{2}, z = \frac{1}{2}y \quad \Rightarrow \quad y_{1,2} = -1;\ \frac{1}{6}$
$\Rightarrow P_1 = (-\frac{1}{2}, -1, -\frac{1}{2}); \quad P_2 = (-\frac{1}{2}, \frac{1}{6}, \frac{1}{12})$

Aufstellen der Hesse - Matrix: $\quad H := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12y+6 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

$\Rightarrow$ Hesse - Matrix am Punkt $P_1 : H_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

Zur Bestimmung der Art der kritischen Punkte wird die *Sylvester - Methode* verwendet:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & 0 & \frac{14}{3} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & \frac{14}{3} \end{pmatrix}$$

$\Rightarrow$ *Sattelpunkt*

Hesse - Matrix am Punkt $P_2 : H_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

Anwenden der *Sylvester - Methode* ergibt:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & 0 & \frac{7}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & \frac{7}{2} \end{pmatrix}$$

$\Rightarrow$ *Minimum*

b, $\quad D_f(x,y,z) = (2x+y, 2y+x, 6z); \quad D_g(x,y,z) = (2x, 2y, 2z)$

$2x + y + 2\lambda x = 0, \quad 2y + x + 2\lambda y = 0, \quad 6z + 2\lambda z = 0, \quad x^2 + y^2 + z^2 = 1$

1. Fall: $z \neq 0 \Rightarrow \quad \lambda = -3$
$2x + y - 6x = 0 \quad \Rightarrow \quad -4x + y = 0$
$2y + x - 6y = 0 \quad \Rightarrow \quad -4y + x = 0$
$\Rightarrow x, y = 0 \quad \Rightarrow \quad z = \pm 1 \quad \Rightarrow \quad f(0,0,1) = f(0,0,-1) = 3$

2. Fall: $z = 0$
$\Rightarrow y = -2x(\lambda+1), \quad x = -2y(\lambda+1) \quad \Rightarrow \quad \frac{x}{y} = \frac{-2y(\lambda+1)}{-2x(\lambda+1)} = \frac{y}{x}$
$\Rightarrow x^2 = y^2 \quad \Rightarrow \quad x = y \quad$ oder $\quad y = -x$
$\Rightarrow x^2 = y^2 = \frac{1}{2} \quad \Rightarrow \quad (x = \pm\frac{1}{\sqrt{2}}$ und $y = \pm\frac{1}{\sqrt{2}}) \quad$ oder
$(x = \pm\frac{1}{\sqrt{2}}$ und $y = \mp\frac{1}{\sqrt{2}})$
$\Rightarrow f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = \frac{3}{2}$
$\Rightarrow f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

2.1 Fall: $y = 0, x \neq 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow$ Widerspruch
2.2 Fall: $y \neq 0, x = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow$ Widerspruch
2.3 Fall: $y = 0, x = 0 \quad \Rightarrow$ Widerspruch

Aufgabe 4

a, $\int\limits_V (x^2+y^2)\,dx\,dy\,dz = \int\limits_{-L}^{L} dx \int\limits_0^R r\,dr \int\limits_0^{2\pi} d\varphi (x^2 + (\cos(\varphi))^2 r^2 =$

$\int\limits_{-L}^{L} x^2\,dx \int\limits_0^R r\,dr \int\limits_0^{2\pi} d\varphi + \int\limits_{-L}^{L} dx \int\limits_0^R r^3\,dr \int\limits_0^{2\pi} (\cos(\varphi))^2\,d\varphi = \frac{2}{3}\pi R^2 L^3 + \frac{1}{2}\pi R^4 L$

b, Sei $\lambda z = \alpha \quad \Rightarrow \quad = \frac{d\alpha}{\lambda}$

$\int\limits_V x^2 + y^2 + \alpha^2\,dV = \frac{1}{\lambda} \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\vartheta \int\limits_0^R drr^2 \sin(\vartheta)(r^2(\cos(\varphi))^2(\sin(\vartheta))^2 +$

$+ r^2(\sin(\varphi))^2(\sin(\vartheta))^2) = \frac{4}{15}\frac{\pi}{\lambda}R^5 + \frac{4}{15}\frac{\pi}{\lambda}R^5 = \frac{\pi}{\lambda}\frac{8}{15}R^5$

Aufgabe 5

a, $\det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 + 1 \stackrel{!}{=} 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

$\Rightarrow \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \pm i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow (2 \pm i)v_2 = v_1$

Wähle $\mathrm{v}_2 = 1 \quad \Rightarrow \quad v_1 = 2 \pm i$

$B := \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -2+i \\ -1 & 2+i \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2}+i \\ \frac{1}{2}i & \frac{1}{2}-i \end{pmatrix}$

$\Rightarrow e^{At} = B \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} B^{-1} = \begin{pmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & cos(t) - 2\sin(t) \end{pmatrix} =: \Phi(t)$

b, $Ay_c = b \quad \Rightarrow \quad y_c = -A^{-1}b \quad \Rightarrow \quad y_c = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c, y_c stellt eine Lösung des inhomogen Gleichungssystems dar:

$\Rightarrow \quad \mathrm{y}= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} c_1(\cos(t) + 2\sin(t) - c_2(5\sin(t)) \\ c_1\sin(t) + c_2(\cos(t) - 2\sin(t)) \end{pmatrix}$

$y(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \Rightarrow \quad c_1 = 2 \text{ und } c_2 = 1$

$\Rightarrow \quad y = \begin{pmatrix} 2 + 2\cos(t) - \sin(t) \\ 1 + \cos(t) \end{pmatrix}$

Aufgabe 6

Lösung des homogenen Teils der Differentialgleichung:

$\frac{dy}{dx} = \frac{x}{1-x^2}\,y(x) \quad \Rightarrow \quad \frac{dy}{y} = \frac{x}{1-x^2}\,dx$

$\int \frac{x}{1-x^2}\,dx = -\int \frac{x}{x^2-1}\,dx = y - \frac{1}{2}\ln(x^2-1) + C \quad \Rightarrow \quad y = \frac{1}{i\sqrt{1-x^2}}\,c(x)$

Variation der Konstanten ergibt:

$y'(x) = \frac{x}{i(1-x^2)^{\frac{3}{2}}}\,c(x) + \frac{1}{i\sqrt{1-x^2}}\,c'(x) = \frac{x}{i(1-x^2)^{\frac{3}{2}}}\,c(x) + \frac{1}{1-x^2} \quad \Rightarrow \quad c'(x) = \frac{i}{\sqrt{1-x^2}}$

$\Rightarrow c(x) = i\arcsin(x) + K \quad \Rightarrow y(x) = \frac{1}{i\sqrt{1-x^2}}(i\arcsin(x) + K) \quad \stackrel{y(0)=0}{\Rightarrow} \quad y(x) = \frac{\arcsin(x)}{\sqrt{1-x^2}}$

Aufgabe 7

Parametrisieren von M :
$\alpha :]0,\infty[\times]0,2\pi[\quad \to \quad]-\infty,\infty[\times]-\infty,\infty[\times]0,1[$
$\alpha : (r,\varphi) \quad \mapsto \quad (\cos(\varphi)r, \sin(\varphi)r, e^{-r^2})$

$d\alpha_1 = -\sin(\varphi)rd\varphi + \cos(\varphi)dr$
$d\alpha_2 = \cos(\varphi)rd\varphi + \sin(\varphi)dr$
$d\alpha_3 = -2re^{-r^2}dr$
$d\alpha_1 \wedge d\alpha_2 = -sin(\varphi)^2 rd\varphi \wedge dr - \cos(\varphi)^2 rd\varphi \wedge dr$
$d\alpha_2 \wedge d\alpha_3 = 2\cos(\varphi)r^2e^{-r^2}dr \wedge d\varphi$
$d\alpha_1 \wedge d\alpha_3 = 2\sin(\varphi)r^2e^{-r^2}d\varphi \wedge dr$

$$\Rightarrow \quad \int_M \omega = \int_0^\infty \int_0^{2\pi} (2r^2e^{-r^2} \underbrace{(\cos(\varphi)^2 - \sin(\varphi)^2)}_{\cos(2\varphi)} - re^{-r^2})drd\varphi = 0 - \pi = -\pi$$

Aufgabe 8

a, Parametrisieren von U :
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2, \quad \gamma : (a,b) \mapsto (a,b) \quad \Rightarrow \quad d\gamma_1 = da, \quad d\gamma_2 = db$
$d\alpha = (-\sin(y) + 101xy(x^2+y^2)^{\frac{99}{2}})dy \wedge dx + (\cos(x) + 101xy(x^2+y^2)^{\frac{99}{2}})dx \wedge dy = (\sin(y) + \cos(y))dx \wedge dy$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} (\sin(b) + \cos(a))da\,db = \int_{\partial U} \alpha = \int_0^{\frac{\pi}{4}} [a\sin(b) + \sin(a)]_0^{\frac{\pi}{4}}\,db = \frac{\pi}{4}$$

b, $dg = 0 \Rightarrow \alpha = 0 \Rightarrow \int_{\partial U} \alpha = 0$